Classification and Mathematical Thinking: Tinkering with Classification Games in a Constructionist Environment

Chronis Kynigos[1], Marianthi Grizioti[2] and Maria Latsi[3]

1. ***Corresponding author**, kynigos@eds.uoa.gr, Educational Technology Lab, National and Kapodistrian University of Athens, School of Philosophy, Department of Educational Studies. Athens, Greece. ORCID: 0000-0002-7412-0422
2. mgriziot@eds.uoa.gr Educational Technology Lab, National and Kapodistrian University of Athens, School of Philosophy, Department of Educational Studies. Athens, Greece. ORCID: 0000-0001-9670-0359
3. mlatsi@ppp.uoa.gr Educational Technology Lab, National and Kapodistrian University of Athens, School of Philosophy, Department of Educational Studies. Athens, Greece. ORCID: 0000-0003-0659-5407

Abstract
Classification is a complex process that involves scientific concepts and higher-order mental processes, such as abstraction, generalisation and pattern recognition. Even though it is an important competence for understanding the world, dealing with data and information, and solving complex problems, the education system embeds just its simplest operations and only in very early schooling. This study examines six middle-school students' activity as they play, modify and redesign two Tetris-like classification games on the mathematical concepts of number sets and angle in an on-line authoring system called Sor.B.E.T (Sorting Based on Educational Technology). The qualitative data analysis of students' dialogues aimed to bring in the foreground the classification processes students applied and the way these processes were entangled with the development of meanings and ideas on the mathematical concepts embedded in the games. According to the results, the play and modding of the two classification games enabled the development of higher-order classification processes, such as objects' properties comparison, properties discrimination and classes' encapsulation. They also supported meaning-making processes and triggered discussions about abstract mathematical notions, such as the concept of angle in various typical mathematical or physical contexts and the concept of number sets, the boundaries of each one and the relationships among them through exploration and learner-generated exemplification.

Keywords
Meaning making, Mathematical thinking, Classification, Constructionist environments, Authoring systems, Game-based learning, Number, Angle.

Classification as a Field for Mathematization

In this article, we introduce ‘Sor.B.E.T’ (Sorting Based on Educational Technology), a digital-authoring system with which learners and teachers can design, create, play and modify Tetris-like classification games. We start by suggesting that, despite a lacking presence in education systems, classification is a competence which importantly requires mathematical thinking and is becoming increasingly pertinent in cultivating digital citizenship. We then consider connections between classification and what, for many learners, are illusive hard-to-grasp ideas in mathematics curricula. We ask: what kind of insights can learners have when they engage in activities to classify, e.g. numbers or angles? Can experience with classifying numbers or angles provide young learners with insights into otherwise obscure aspects of such mathematical concepts? What meanings do students generate about classification and what meanings about number or angle?

In the literature, classification has received several definitions and approaches. Many researchers describe it as a logical–mathematical operation necessary for developing formal operational thinking (Inhelder & Piaget, 1964; Adey & Shayer, 1994). In this approach, it is seen mainly as a by-product of other mathematical notions that is achieved through empirical exploration and the use of individual aspects of classification through traditional logical–mathematical problem solving (Milne, 2007). Furthermore, many researchers see classification as a key operation of learning and understanding the world, as it enables children to respond rapidly to new experiences by applying known patterns and categorising objects (Owen & Barnes, 2021).

Similar to computational thinking, problem solving and critical thinking (Grover & Pea, 2018, Wing, 2011), classification involves concepts, such as logical operations, classes and objects, but also thinking processes like mathematical reasoning, abstraction of rules, generalisation, creation of classes and patterns, design of the classification system. Developing and using such processes is considered a difficult task for students, since they are quite abstract and can hardly be described with formalistic rules or tangible representations (Armoni, 2013; Robins, Rountree & Rountree, 2003). As a result, educational systems tend to leave them out of K–12 curricula and only include basic classification concepts in very early schooling (Milne, 2007).

As a result, the majority of classification-related studies focus on testing students’ performance, while practising basic operations of classification through quizzes and diagnostic tests which involve sorting tasks including verbal or schematic material (Micklo, 1995; Mathy & Bradmetz, 2011; Kurbanova & Salikhova, 2016). At the same time, the few existing digital tools and activities that address classification aspects for education implement quite strict designs, such as closed tasks and simplified exercises that deal only with basic classification processes. These activities do not enable students to experiment with and explore logical operations of classification in a problem-solving, multi-disciplinary context.

Recently, however, some researchers have started to reconsider classification as a key competence for children to develop concepts and meanings for different scientific domains and a core aspect of 21st-century skills, computational thinking and information literacy. Core classification processes, such as comparison of objects and data, discrimination between properties and generalisation of characteristics from classes are part of computational thinking and information literacy, two of the most highlighted and studied competencies for the 21st century (Vuorikari, Kluzer & Punie, 2022). Researchers have highlighted the importance of students developing computational thinking throughout all levels of education and across the curriculum to succeed in 21st-century society (Grover & Pea, 2018; Voogt & Roblin, 2012).

Information literacy is considered a core element of digital competencies that all 21st-century citizens should acquire. Thus, in this context, classification is being approached as an inter-disciplinary, meta-subject practice that involves high-order logical operations and computational practices (Cao, Kurbanova & Salikhova, 2017; Krnel, Glažar & Watson, 2003). However, there is still a lack both of theoretical understanding of student learning processes regarding classification processes and of technological means that would enable them to express and develop such learning.

We suggest that the wake of this attention to classification as an important citizen competence should enhance rather than distract attention to the connections between classification and mathematical thinking. We thus focus our discussion on the mathematical nature of classification activity, also considering it as a vehicle potentially to gain special insights into traditional mathematical concepts, such as number and angle. We approach these two layers of meaning-making (classification and mathematical concepts) as intertwined, considering what classification skills could be cultivated in the process of classifying mathematical concepts.

In designing Sor.B.E.T, we wanted to provide teachers and learners with a tool with which to construct classification games, leaving the field of instances and classes transparent and up to its users to define. Game modding, i.e. tinkering with the rules of a game, is seen as a constructionist process (Papert, 1980; Kynigos & Grizioti, 2020) that offers low threshold and high ceiling (Resnick & Silverman, 2005), lowers the stakes and enables students to tinker with the game content and rules, discuss ideas about them and express personal meanings on scientific concepts.

Thus, the presented study aims to answer the following research questions though a constructionist theoretical perspective:

a) Which classification processes do students apply while they play and modify two classification games on the mathematical concepts of number sets and angle in Sor.B.E.T digital environment?
b) How do these processes enhance the development of meanings and ideas for those concepts embedded in the games?

Theoretical Framework and Related Work

Constructionist mathematical meaning-making

Sor.B.E.T was designed as an authoring system for constructionist learning processes. Sor.B.E.T artefacts are, essentially, Tetris-like classification games. Constructionism, based on Papert's generic vision and ideas for learning (Papert, 1980), claims that learners put concepts into use and generate powerful ideas through the processes of tinkering, sharing and discussing over personally meaningful artefacts through programmable digital media (Ackerman 2001; diSessa, 2001; Kynigos, 2015, 2020). Constructionism has successfully been used as a learning theory and as a design framework for enhancing the generation of student meanings and "situated abstractions" (Noss & Hoyles, 1996) about scientific concepts through programming.

Papert's original focus was on the kind of mathematical thinking processes which could be enabled and nurtured in a constructionist environment, where the artefacts were figural models created by means of programming with a mathematical language (Papert, 1980). The studies that ensued were understandably about the nature of the process and meanings generated by learners (Noss & Hoyles, 1996). This brought about a reaction regarding: a) what mathematical ideas were actually learned during a constructionist activity; b) to what extent and how these ideas could be connected by learners to their corresponding abstract form and formalisation as they appear in traditional curricula and exam questions (Geraniou & Mavrikis, 2015).

In this article, we claim that constructionism can become a powerful tool in the 21st-century competence-oriented mathematical pedagogies (Geraniou & Jankvist, 2019), and at the same time allow for meaning-making regarding mathematical concepts and objects. This is why we ask: what is mathematical about classification and, at the same time, what insights could classification of numbers and angles provide to the concept of number and angle respectively?

Problems in number and angle related to classification

Viewing the learning of mathematics as a constructive activity, we want to encourage students to investigate the properties of the various number sets and the relationships among them, as well as to investigate the concept of angle in various typical mathematical and physical contexts through the classification of examples and through learner-generated exemplification (Watson & Mason, 2006). Young learners' difficulties with understanding aspects of number have been an important focus and one of the founding issues in mathematics education research (see the seminal work of Hughes, 1986, as a typical example).

According to Freudenthal (1973) there are many tricky number concepts both as regard content and form. Regarding content, numbers can be classified into sets, called number sets, such as natural numbers, integers, rational and real numbers. Each of these number sets is a subset of the next one. So, for example, a natural number is also a rational number, and every rational number is also a real number. In other words, the various number sets are connected to each other by embeddings; adjunctions lead to larger and restrictions to smaller number sets.

Many of the definitions of the various sets of numbers refer to representations. Discriminating whether a number belongs to a given set is based on whether or not it can be represented in a given form. For instance, a rational number is a number that can be expressed as a fraction with an integer numerator and a positive integer denominator. So far, there is extensive research on young children's number sense (Hughes, 1986), as well as on the way students understand number representations within specific number sets: for instance, representations of natural numbers (Zazkis & Gadowsky, 2001) or representations of rational numbers (Psycharis, Latsi & Kynigos, 2009), but not on the way students come to be aware of the boundaries of each number set and the relationships among them.

Students' perception of number sets is usually restricted by their everyday school practices, which emphasises calculations rather than on the fundamental properties of each number set (Zazkis & Gadowsky, 2001). In our study, we gave learners a Sor.B.E.T game involving the classification of numbers in number categories, such as integer, fraction, decimal, etc. We did this intentionally, hoping the learners would soon start questioning the rules of the game, noticing that these categories were either not distinct but embedded in one another, or referred to different representations of numbers (for instance, a fraction can be represented as a decimal and vice versa).

Angle is another one of those concepts being at the centre of curricula in primary and early secondary education, and yet causing a lot of confusion among learners. According to Henderson and Taimina (2005), angle can be defined from at least three different perspectives: (a) angle as a directional relationship between two geometric shapes, i.e. formed between two geometrical objects which can be either segments or 2D geometrical figures; (b) angle as a dynamic notion, indicating a change of one direction both as a turn and as the result of a turn; (c) angle as a measure represented by a number. In typical school education, angle is approached as a static geometric figure (Freudenthal, 1983), disconnected from real-world contexts.

The convention to represent angle as a loosely positioned arc joining two semi-straights with a joint starting point disorientates learners in different ways, a central one of which is that the talk is about the figural object – instance generated each time it is referred to (Clements & Burns, 2000; Mitchelmore & White, 2000). Students develop an abstract idea of what an angle is and have difficulties in transferring that knowledge out of their textbook representations (Latsi & Kynigos, 2022; Kynigos & Psycharis, 2013).

Even though angle (along with length and distance) is one of the most important mathematical tools in a wide variety of physical situations, these physical situations are not easily correlated or connected to angle concepts by children at the end of primary school (Freudenthal, 1983; Mitchelmore & White, 2000). Digital media seem to provide the potential to relate the concept of angle explicitly to children's physical experiences, and re-address the use of various representations of physical contexts (dynamic or static), as well as of more typical mathematical representations for students to form meanings about the angle.

Here, we report on how students' intuitions and ideas concerning the angle concept were challenged while trying to classify the different kinds of angular representations according to their estimated angular measure. Asking students to identify and classify angles in various physical and typical mathematical contexts according to their measure, as well as to exemplify while modding a classification game, might encourage students to: a) recognise similarities between different angle contexts, which is considered as a prerequisite to angle measurement using a protractor; b) co-ordinate different aspects of the angle concept, which is conducive to meaningful teaching of abstract definitions of angle (Mitchelmore & White, 2000).

Play and design classification games for mathematics

Sor.B.E.T (Sorting Based on Educational Technology) is an on-line application of Educational Technology Lab (NKUA) that allows playing and designing classification games and it is freely available on-line (Grizioti & Kynigos, 2023). The prototype version was developed as part of a Master thesis (Giama, 2020), and it was further developed to its current version including also block-based programming. By using Sor.B.E.T, learners can engage in constructionist activity interchangeably playing, modifying and designing Tetris-like classification games with diverse content and complexity. Students can easily switch between the roles of players and designers, question the game content and express their own ideas through modding.

According to recent studies, game modding seems to be a promising approach for the development of higher-order skills, such as computational thinking, system analysis and design thinking, since it scaffolds student engagement with the learning content, gradually transforming their roles from user to creators (Kynigos & Grizioti, 2020; Grizioti & Kynigos, 2021; Örnekoğlu Selçuk et al., 2022). In a Sor.B.E.T game, the player scores by 'pushing' elements falling off the top of the screen to drop into the right category box at the bottom (Figure 1). 'Pushing' elements can be done by picking and dragging on a screen. When an element falls into a box, the box changes colour (red or green) providing feedback on the correctness of the classification. Moreover, when the game is over, the players can access, download and reflect upon their classification decisions through the game log that shows in a graphical way what they classified into each box.

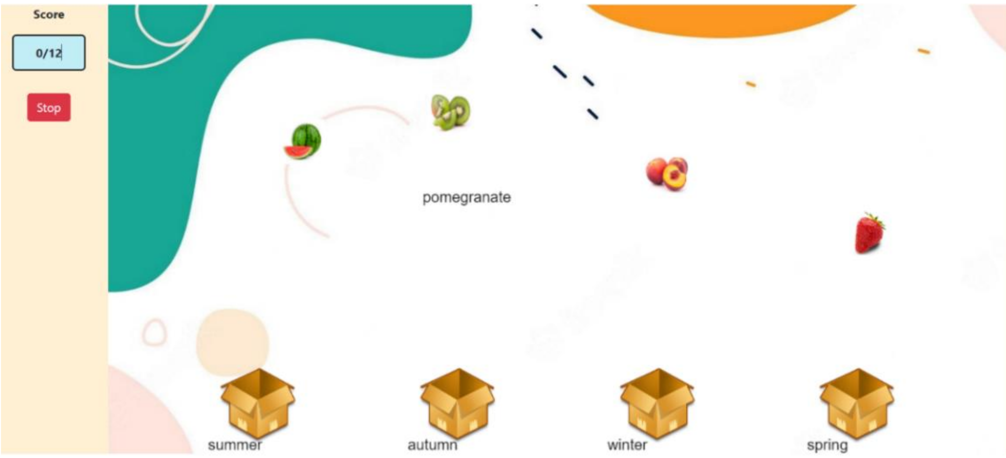

Figure 1: *Screenshot of the Sor.B.E.T environment in Play Mode: player classifies falling objects (images or text) to the category boxes*

The 'Design Mode' offers two interconnected affordances for modifying or creating new classification games, i.e. an interactive database and block-based programming. The database represents the objects as rows and the categories they belong to as columns (Figure 2). The users can easily add or remove game objects in the database rows. An object can either be an image of any format or a text, offering multiple representations of the same concept. The users can also determine how many instances of the same object will fall and in which category(ies) they belong. They can change the number of categories (game boxes) by adding or removing columns, and modify their name on the top of the column.

Sor.B.E.T follows the classification model of 'one to many', which means that one object could be classified into more than one category. This design decision aims to raise discussions among players about the intersections or mutual exclusions of available categories based on the object's properties. Sor.B.E.T also supports programming as a means of self-expression through the game (Kynigos, 1995). Block-based programming allows the designer to modify some game mechanics, such as the falling speed and the density of the falling objects based on certain events.

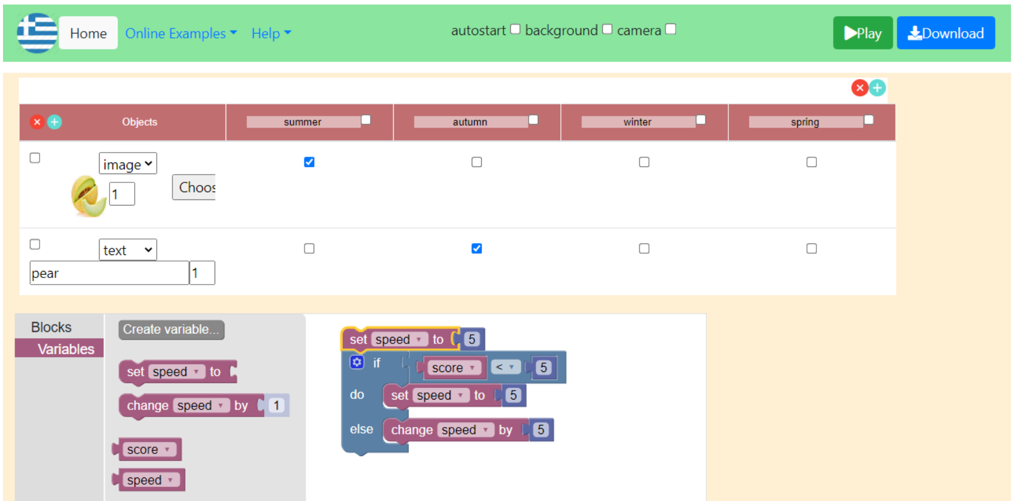

Figure 2: *Sor.B.E.T block-based programming in Design Mode: the user can modify the game mechanics, like the game speed, with specialised blocks*

For the purposes of this study, we designed two mathematical classification games in Sor.B.E.T. The rational for having two games was: (a) to be able to study the development of student classification processes in different contexts (RQ1); (b) to gain insight into students' mathematical meaning-making while using those classification processes (RQ2). Both games were designed to bring into the foreground the notions of union, intersection, difference and exclusivity of game classes, based on the recognition and analysis of their common or unique properties.

The first, called 'Classes of Numbers' (Figure 3), is a game focusing on the classification of falling numbers into classes of the decimal numerical system, i.e. Real, Rational, Irrational, Fractional, Positive, Negative, Integer, Natural, Decimal. The categories were chosen so that some classes encapsulate others. For instance, the integer class includes objects (numbers) that also belong to positive and negative classes; the rational class encapsulates fraction and integer classes and, as a consequence, some objects from the positive and negative classes. The game aims to engage students with the numerical sets, which are usually taught through abstract and disconnected rules, through a tangible, familiar and relatable representation.

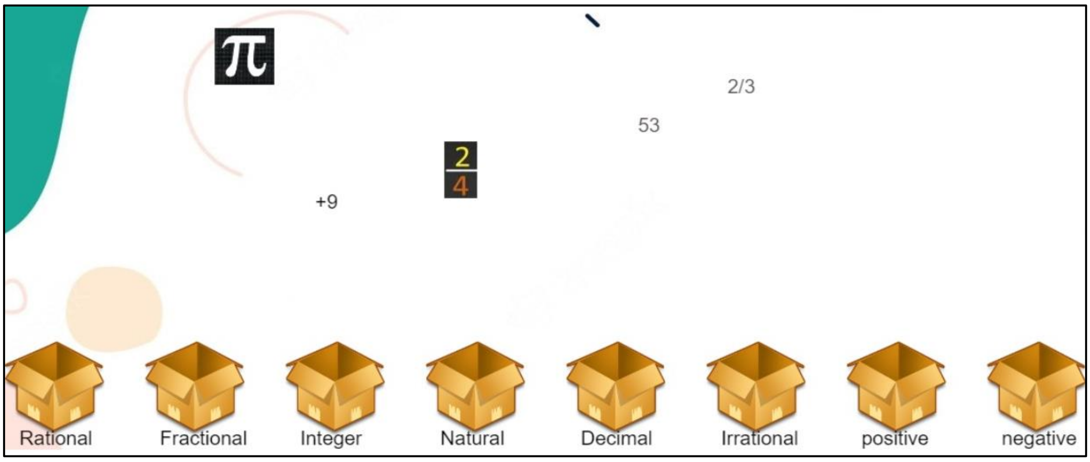

Figure 3: *The 'Classes of Numbers' game: the player classifies numbers to classes of the numerical system*

The second, called ‘Falling Angles’ (Figure 4), is a mathematics game focusing on the concept of angle. The game approaches the notion of angle through different representations drawn from real-life contexts, in contrast to the traditional abstract representations in school textbooks. In this game, the player has to classify falling pictures or text representing objects in an angle, e.g. clock hands, bird wings, time, hands position, to five angle categories (acute, right, obtuse, straight, non-reflex). Some pictures can be categorised in more than one category, since they depict more than one angle. This aims to raise discussions about the concept of angle among the students who collaboratively play and modify the game.

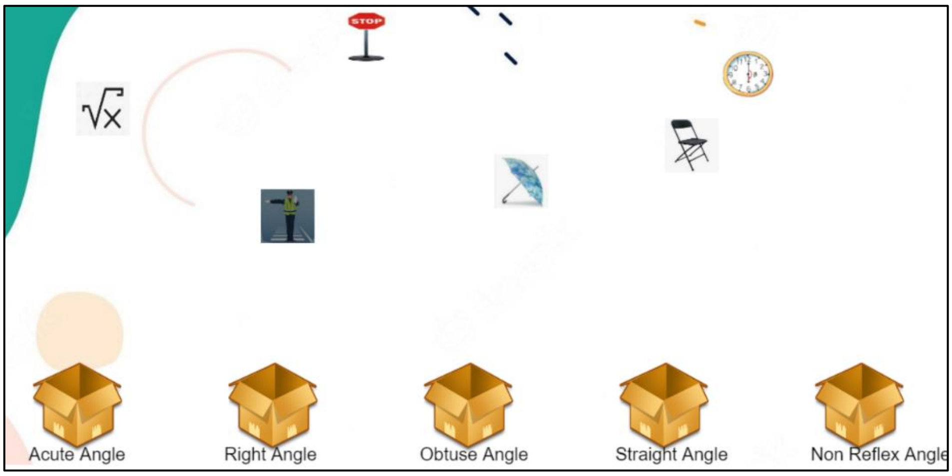

Figure 4: *The ‘Falling Angles’ game: the player classifies pictures according to the angles they depict*

Research Method

The study presented in this article is the first part of an on-going, design-based research (Bakker, 2018; Barab & Squire, 2004). Design-based research entails the ‘engineering’ of tools and task, as well as the systematic study both of the process of learning and of the means of supporting it (Gravemeijer & Cobb, 2006), in order to create new theoretically expressed understandings about areas for which little is known. In this article, we present the design of classification games and tasks, and we study the development of students’ classification processes and mathematical meaning-making. Its results are going to inform the redesign of games, tasks and theory in the next research cycles.

Context and participants

The study took place in an experimental middle school in Greece as part of an after-school math club, where a school math teacher and a researcher were present. It had duration of four hours divided into two two-hour sessions. The participants were six students, four boys and two girls, aged 13–14 years old. The students considered themselves regular or light gamers, but they had no experience with game design.

Regarding mathematical concepts, students had been taught number sets in their school mathematics courses, but they were not accustomed to them as an integrated system. They had basic knowledge of angles in the context of geometry that did not include the reflex angle. Two weeks before the study, all students were informed about the study context, purpose, data collection and analysis processes with a written announcement and they voluntarily expressed participation interest. All participants and their parents gave written consent.

Activities and study process

For designing the study activities, we employed the approach of game modding, in which students first play and then modify elements of an existing game, in order to create a different version of it, called 'mod' (Sotamaa, 2010). The participants worked in three Groups of two students each, that were formed randomly by the mathematics teacher. In each two-hour session, students were engaged with one of the games using a lap-top. They initially played each game several times and they then modified it. Each session was divided into three parts. In the first part, each team played the game several times. In the second part, they modified the game creating a new version of it. In the third part, each team played the game of another team giving feedback.

Data collection and analysis

The researchers collected three types of data resources throughout the study. These included screen and audio capturing of each Group with HyperCam 2.0., student modified games and researcher's observation notes. In order to develop a deep understanding of students' learning and meaning-making processes, the researchers performed a qualitative analysis of the collected data. First, they transcribed the audio recordings using anonymization techniques. The transcriptions were then correlated with screenshots and observation notes to provide a complete picture of student activity. Then two researchers analysed and coded the transcribed recordings using the 'critical incident' as the analysis unit, i.e. a representative moment of student activity relevant to the questions of the study (Tripp, 2011). The incidents usually consisted of a dialogue combined with data from screen capturing, but it could also be a single student utterance while interacting with the tool.

For coding the incidents, the researchers followed a bottom-up, qualitative thematic analysis (Braun & Clarke, 2006), starting with open coding and continuing with repeated cycles of coding comparisons, clustering and merging until reaching the final themes. In Table 1, we present an overview of the process that has been followed: specific codes were initially used to label critical incidents. These codes were progressively Grouped in concepts, then in more abstract categories and finally in two main themes

Table 1: *Analysing data: moving from codes to main themes*

Codes	**Concepts**	**Categories**	**Themes**
Objects' properties comparison, Pattern recognition, Categories comparison	Identifying data categories	Application of classification processes	Classification and meaning-making processes while playing the game
Categories intersection, Categories exclusion,	Identifying/Designing category properties		

Categories completion, Categories Encapsulation-Inclusion			Classification and meaning making processes while modding the game
Classification criteria, Abstraction of class Properties, Generalisation, Classification model design	Analysing/Designing the classification model		
Number sets, Number content, Number form, Rational/Irrational, Fraction, Integer/Decimal	Generating meanings about the mathematical concept of angle	Mathematical meaning-making	
Reflex/non-reflex angle, angle in physical/typical contexts, straight angle, full angle, acute/obtuse angle	Generating meanings about the mathematical concept of number		

Results and Discussion

The analysis of our results is organised around two main themes that bring into the foreground core issues related to students' classification processes, which, at the same time, supported meaning-making processes and triggered discussions about number sets and angles: a) classifications and meaning making processes while playing the games; b) classification and meaning-making processes while modding the games. However, in the analysis of the results presented here, we focus only on: a) classification processes and meaning-generation about number sets while playing the 'classes of numbers' game; b) classification processes and meaning-generation about the concept of angle while modding the 'falling angles' game. We opted for this categorisation of the presentation of our results, since it would allow us to focus and deepen our analyses on the critical issues that have emerged while answering our research questions in the limited length of an article. Since, in most cases, the classification processes co-existed with the meanings that students developed, we discuss incidents that respond to both questions, rather than examining them separately.

Classification and meanings generation: Meanings about number sets while playing 'the classes of numbers' game

The analysis of critical incidents showed that all Groups, when they initially played the game, tended to categorise each number into one category, leaving some categories with no classified objects. In fact, two of the Groups realised that, "the game has more categories than the ones the numbers belong to", while the most common categories that students initially chose to classify the numbers were, “Positive, Negative and Decimal”.

However, as students kept playing the game, they started discussing whether one number could belong to more than one category, changing their initial classification criteria and experimenting with pushing the falling numbers to different boxes. Thus, they developed a new classification process coded as ‘Objects’ properties comparison’. This code was used in critical incidents where students compared the mathematical properties of the falling objects to decide on their classification. This process was usually developed as students played the game repeatedly trying to discover all the possible classes each number belonged to.

Table 2: *Critical incident 1: Group 2 students discuss about the categories to which a number belongs*

line	Alias	Transcript
1	George	Put 6.99 into the rational box.
2	Jake	But it’s not an integer.
3	George	It doesn’t matter. It is rational and decimal.
4	Jake	But all numbers that were taken as correct in the rational box were integers.
5	George	No. 2/3 was also correct. I think that all integers are rational, but also other types of numbers are rational as well, such as fractions. Try and see.

In critical incident 1 (Table 2), students of Group 2 play the game for the sixth time and they have started exploring the different classes to which each number belongs. There is a disagreement between the two students on whether 6.99 is a rational number, leading to a discussion on which other number representations are included in the “Rational” number set. George explains to Jake that all integers are rational, but this does not mean the opposite, i.e. that all rational numbers are necessarily integers. His explanation is based on personal free exploration of the game, observation, formulation of conjectures, testing and conclusions, following the process of ‘Object Properties Comparison’ and discussion of the ‘Classification Criteria’. These processes led him to express a complex mathematical idea, that of numerical sets and their encapsulation, not through formalism but through something tangible and relatable to him, i.e. the game.

In critical incident 2 (Table 3), which took place just after critical incident 1, the two boys continue their mathematical reasoning about the encapsulation of numerical sets. Jake makes an incorrect assumption that all numbers are rational, and George corrects him by noticing that there is the ‘irrational’ category which excludes rational numbers. Thus, some numbers must belong to one and not the other category. Again, by comparing the game results so far (Objects properties comparison) and excluding integers, decimals and fractions, George suggests that root 2 could be an irrational number.

Table 3: *Critical incident 2: Group 2 express some general rules for the number sets*

Line	Alias	Transcript
1	Jake	You were right! So, all numbers are rational? This is very easy for the game. We can beat it just like that.
2	George	But then why does it have the box ‘irrational’? Some of them must be irrational.
3	Jake	What does this mean?
4	George	I don’t know. Let’s search for a number that is not integer, nor decimal or fraction.
5	Jake	Oh, oh, square root of two?

The analysis also revealed that students initially approached the concept of number as an amount representation of something tangible, as the number of elements of a set without further properties. In that case, students classified each number only to one category, according to their intuitions about that representation (e.g. 2/3 represents a fraction), assuming that it cannot belong to another one (classification criteria). This meaning is quite strict and may lead to misconceptions about the idea of number. As playing went on, students changed their classification criteria progressively approaching number as having both content and form, i.e. as an amount representation in a given form (e.g. rational, positive), a form that is shared with other numerical representations belonging to the same number set (incidents coded with both codes ‘Number Content’ and ‘Number Form’). This meaning requires a higher level of abstract thinking and is considered a complex mathematical idea, difficult for young students to grasp (Hughes, 1986).

The analysis also showed that, as the activity progressed, the critical incidents in which students perceived number as having both ‘content and form’ significantly increased while those focusing on the ‘amount representation’ of number decreased. For instance, in critical incident 3, students of Group 1 are playing the Numbers game for the fourth time.

Table 4: *Critical incident 3: Group 1 students discuss about the number as an object and as a representation*

Line	Alias	Transcript
1	John	Ok, so 6/3 is a fraction, we know that.
2	Chris	Wait, wait. What else can it be? Positive for sure. Make it fall to the positive box.
3	John	Oh, and maybe it is also an integer since 6/3 equals 2.
4	Chris	Let’s try it next time. If that’s correct, this would mean that 2/4 is also decimal?!

While in the beginning John sticks to the representation of the number 6/3 as a fraction with a certain numerical content, Chris starts thinking of 6/3 as a special form, as a representation with certain properties. Then, both students are engaged in more advanced mathematical thinking, trying to express the properties of this numerical representation according to the properties of the game’s classes. They get involved in an interesting mathematical conversation on whether 6/3 is also an integer and 2/4 a decimal, which is quite advanced regarding their age and the level of their mathematical knowledge. Even though their assumption is not correct in formal mathematical terms, the game engaged students in a thinking process of comparing the two representations, e.g. 2/4 and 0.5, and discussing their numerical content and their classification to the number sets of the game. It is also interesting that students do not refer to formal definition, but rather try to find out the properties of each number representation working with examples and non-examples.

Since the games followed the one-to-many classification model, students developed the classification process of 'categories comparison'. This code describes incidents of comparing the game's categories, in order to identify whether and how they complement, intersect or exclude one another for certain number properties. While playing the game, they tried to discover whether the falling objects belonged to one, more or all the categories, as in critical incident 3 (Table 4). Students generated meanings about number sets through exploration and experimentation while playing a 'tangible' classification game, rather than learning 'by heart' abstract formalistic rules illustrated by diagrams. In Table 5, some students' utterances are presented, that show classes inclusion (line 1), exclusion (line 2, line 3) or intersection (line 4).

Table 5: *Selected utterances from 'classes of numbers' game*

Line/Group	Critical Incident Utterance (s)
1/Group 1	"A positive or negative number might also be an integer."
2/Group 1	"Wait! All fractions are also positive or negative numbers."
3/Group 2	"Some categories automatically exclude the others. For instance, an integer cannot be a decimal and a negative cannot be a positive. But can a decimal be a natural?"
4/Group 3	"Oh, look! The negative integers are not natural! So, it's not correct that all integers are natural." "Oh yes! Only the positive ones!"

Modding a classification game like 'the classes of numbers' game could offer more chances for experimentation, meaning-generation and rather more advanced classification operations, such as the creation of new classes. In the following sub-section, we focus on classification processes and meaning generation while students are modding the 'falling angles' classification game.

Classification and meanings generation: Meanings about angles while modding a classification game

After playing the 'falling angles' game several times, students were asked to modify it. Initially, they have thought of adding new objects. In critical incident 4 (Table 6), Group 3 students are thinking of adding a new object that belongs to two categories, viewing the acute angles as a sub-section of the non-reflex ones. Then, after a researcher's prompt, they are trying to find an object that does not belong to any category of the game, and they decide to add to their game, as a new object, a circle and a full-stop. This way they are rather adding their game non-examples or a category elimination task that sharpens the distinction between the categories of the game.

Table 6: *Critical incident 4: Group 3 students discuss how they could modify the angles game*

Line	Alias	Transcript
1	Researcher	Can you find an object that belongs to two categories?
2	Mary	A clock showing twelve hours and one minute.
3	Researcher	Where does it belong?
4	Mary	Acute and non-reflex
5	Researcher	Can you find an object that does not belong to any category?
6	Mary	Not to any category?
7	Helen	The circle?
8	Researcher	Why?
9	Helen	Because it does not have any angles!
10	Mary	Or a full-stop!

While modding the ‘falling angles’ game – after playing it several times – students seem to get a deeper understanding of the concept of angle, and of the validity and value of their conjectures. Adding new objects from various physical – static, such as the chair representation, or dynamic, such as the football player running – contexts (see Figure 5 below) while trying to classify the objects’ angles, students are generalising about the kind of angles that a 2D object can have. In this process, they take into account the whole plane and not only the part of the plane that is included between the angle’s rays (generalisation and abstraction processes).

For instance, in critical incident 5, Group 1 students, having in mind the complete angle that corresponds to the central angle of an entire circle, conclude that all objects that have an acute or an obtuse angle have also a reflex one (line 4), following the process of properties abstraction. A reflex angle complements an acute or an obtuse one. They also think of the special case of the straight angle (line 2), where there is no reflex angle, which is another counter-example generated by the students that restricts the scope of their generalisation process.

Table 7: *Critical incident 5: Group 1 students’ generalisations about the concept of angle while modding the angles game*

Line	Alias	Transcript
1	Chris	So, all angles have a reflex angle from the other side.
2	John	Not all. The straight angle is 180 degrees, so it doesn’t.
3	Chris	Yes, you are right.
4	John	All objects that have one acute or one obtuse angle also have a reflex one.
5	Chris	Sir! We discovered a new rule!

Through playing and modding the ‘falling angles’ classification game students identified angles (static or dynamic) embedded in different kinds of figures (see Figure 5 and 6), either typical mathematical or related to their physical angle experiences, where the arms of the angle were not always clearly visible. In critical incidents 4 and 5 (Tables 6 and 7), students approach angle not just as the union of two rays with a common end-point and the region contained between the two rays, but as part of the full circle. In other words, they conceived the angle of trigonometry – according to Freudenthal (1973) classification of the concept of angle – that is viewed as a centre angle of a circle (say, with radius one) and that can be measured by 360^o with a full circle protractor (or by 2π).

While trying to classify new objects chosen by them to the existing categories, students generated non-examples – the straight angle (line 2, Table 7) – to restrict the scope of the existing categories as well as counter-examples – the full stop. These meaning-generation processes were inextricably linked to students' classification processes. For categorising objects according to their angles, students used the 'categories completion' process, conceiving the full angle. This code was used in critical incidents, in which students designed the game categories in a way that some of the data of one category were also part of the data-set of another category.

Figure 5: *Screenshot from Group1 modified game: students have added representations of real-life objects from various physical contexts in order to depict various angular relationships*

Trying to mod the 'falling angles' game, an original and unexpected idea emerged in Group 3: they decided to modify the game's categories, so that the player classifies the falling objects into geometrical shapes' categories, e.g. rectangle, triangle, etc. This modification led to the development of classification processes that were coded as: 'Categories Encapsulation-Inclusion' (i.e. Designing a game category (class) that includes another with the latter inheriting the properties of the larger one) and 'Classification Model Design' (i.e. Designing and setting up rules for the classification model of the game e.g. how many categories it will have).

In incident 6 (Table 8), students try to decide the categories of their redesigned game. Students are engaged in an interesting conversation about the categories they are going to have in their game and whether one category includes or excludes the other. Numerous studies have shown students' difficulties in understanding the inclusion relations between classes of geometrical shapes causes, i.e. between quadrilaterals. These difficulties are related to tacit properties and prototype phenomena (Jones & Tzekaki, 2016). The simple identification of a geometric figure does not necessarily allow students to identify inclusion or disjunctive relations. However, it seems that, while modding 'falling angles', classification game students are engaged in exploring inclusion relations between quadrilaterals, which is considered (Matsuo, 2007) as the highest state of understanding relations among figures.

Table 8: *Critical incident 6: Group 3 students discuss about categories' properties while redesigning the game*

Line	Alias	Transcript
1	Mary	Let's also have rectangle and square categories.
2	Helen	But a rectangle is also a parallelogram, isn't it? … Yes, because by saying parallelogram we include all types of parallelograms even the rectangles.
3	Mary	[pause] But there are so many other parallelograms that belong to this category.
4	Helen	Yes, but all the objects that go to the rectangle category will also go to the parallelogram category.
5	Mary	Oh, oh, you are right. And the square too.
6	Helen	What about having a rectangle and oblique parallelogram? Then we are sure that these two have different objects.
7	Mary	Ok. And we can have a more general one that fits both, to make the game more challenging … How is it called?
8	Helen	What how is it called?
9	Mary	What are a rectangle and a parallelogram and a hexagon, etc.?
10	Helen	Oh! A polygon I think.

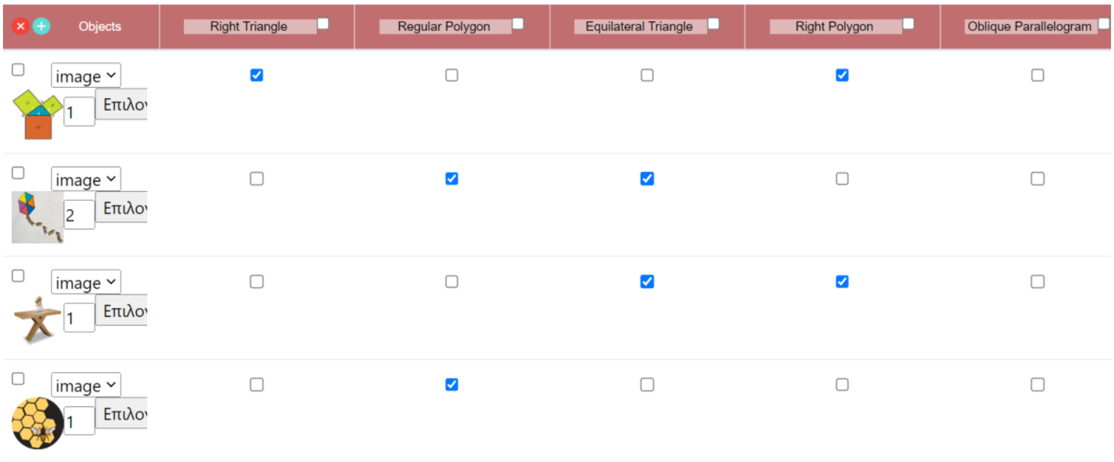

Figure 6: *Group 3 modified game in design mode: students have used pictures from real world that depict more than one geometrical shapes and thus they belong to more than one class*

While designing the classification model students also got engaged with the 'Categories Encapsulation' process. In the critical incident 6 (line 2 and 4), the category 'rectangle' is considered as a sub-category of the 'parallelogram' category, while, in line 6m the category 'rectangle' excludes all the objects belonging to the category 'oblique parallelogram'. Finally, in line 10, students are trying to find out an overarching category that comprises rectangles, parallelograms and hexagons, and they conclude that this category is 'polygon'. After several changes, the final game categories were: Regular polygon, Equilateral triangle, Right triangle, Rectangle, Oblique Parallelogram (see Figure 6). Through classification processes, students engaged in a mathematical process of comparing classes of geometrical shapes, discriminating their properties and generalising while creating sub- and overarching categories.

In an open-ended constructionist task – such as in the case of modding a game, presented above – students usually come up with unexpected ideas that may be out of teachers' didactic intentions (Kynigos, 2015). Although the mathematical concepts students are engaged with are not directly linked to the topic of angle, the previous incident is characteristic of how, while modding 'the falling angle game', students engaged in a mathematical process of creating sub- and overarching categories. In this process, they used classification processes, such as intersection, exclusion and completion, while using mathematical terms and generating mathematical meanings.

Conclusions

This study examined middle-school students' classification processes as they played and modified two mathematical classification games in the digital environment Sor.B.E.T. The analysis of participants' dialogues allowed the researchers to identify processes and meanings students developed throughout the activity. Working with examples and non-examples, students analysed the classification model of each game and developed complex classification processes such as objects' properties comparison, generalisation and categories encapsulation, in order to understand and modify the classification model of the two games. An important role in this process seemed to have been played by the type of Sor.B.E.T classification model (one to many) that urged discussions, i.e. what exactly do I classify, based on which properties, and what are the categories' intersection, completion and exclusivity rules?

In parallel classification processes in both gameplay and design-fostered, meaning-making processes and content knowledge development about mathematical concepts which may have not been easily accessible with traditional means. In our study, we found that students, in order to understand the classification schema and win the game developed meanings for the concept of number, revealing possible misconceptions and issues related to that concept. Through the game, students gradually started to refer to numbers as having both content and form, i.e. as an amount representation in a given form (e.g. rational, positive and decimal), a form that is shared with other numerical representations belonging to the same number set. Moreover, this dual meaning generation for the notion of number fostered advanced mathematical discussions between students like the example of whether 2/4 is the same numerical entity as 0.5 and whether it can be considered as a decimal number. Students became gradually aware of the boundaries of each number set and of the relationships among them, an issue which is rather under-researched so far.

We also observed learners correlating typical and non-typical representations of angles in various physical (dynamic or static) and mathematical contexts, while they were trying to identify and classify angles according to their estimated measure. Through progressive reflection on the classification of the angular relationships between objects while playing the 'Falling angles' game and through learners' generated exemplification while modding the game, angle was conceived not just as the union of two rays with a common end-point and as the region contained between the two rays, but as part of the full circle. All these issues are considered as critical to a meaningful teaching of abstract definitions of angle (Freudenthal, 1983; Mitchelmore & White, 2000).

It is interesting that, while modding the game, students generated their own examples, counter-examples and non-examples of the mathematical concepts embedded into the games. This learner-generated exemplification opened up new discussions and encouraged students to search through the mathematical context from varying points of view discerning features and structures. For instance, by modding the ‘falling angles’ classification game in ‘unexpected’ to teachers’ ways, students came to terms with definitions and properties of quadrilaterals: they explored inclusion relations between quadrilaterals which is considered (Matsuo, 2007) as the highest state of understanding relations among figures. Through classification processes students engaged in a mathematical process of comparing classes of geometrical shapes, discriminating their properties and generalising while creating sub- and overarching categories.

Through the critical incidents presented in the results, we can see that student classification processes and meaning-making processes were fostered by two factors: a) the classification game and the Sor.B.E.T affordances, i.e. in order to score higher students had to discover the classification model and the math rules behind it; b) the modification of the game and the Sor.B.E.T affordances, i.e. students exemplification through the creation of classes and the definition of objects’ categories in the database.

The results of our research point to the value of classification and game modding for making abstract mathematical ideas learnable, usable and meaningful to learners. It seems that the classification games lowered the stakes of formalistic mathematics, enhancing learning through productive failure, exploration, self-expression and discussion of scientific concepts. Of course, we hasten to point out that we are not suggesting that classifying numbers and angles will lead to a comprehensive understanding of those mathematical concepts. In this article, we are just investigating whether there are aspects of these concepts which could be clarified through classification, and, in that case, which ones. Similar activities and games could be developed and studied integrating classification as an inter-disciplinary process that can leverage 21st-century skills development, like computational thinking and information literacy across the curriculum.

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Conflict of Interest

There is no conflict of interest in the current research. The software (SorBET) that is used is open source and freely available.

Data Availability Statement

The transcribed data of audio recordings and interviews have been decoded so that there are no data on student identity and they are stored in secured servers of the NKUA. Parts of them can be provided after a request and a justification of use to the authors. The data are currently available in the Greek language, but parts can be translated if necessary.

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