ABSTRACTION OF MOTION IN PATTERNS ENHANCE ALGEBRAIC GENERALIZATION IN MATHEMATICS CLASSROOMS

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In this paper we present results of a study that investigated year 8 (ages 12-13) students' construction of meanings for algebraic generalisation. The students worked in groups in the classroom using a specially designed microworld (eXpresser) to explore a task that linked the realistic situation of a bee hive to figural patterns. In order to trace student's meaning making for algebraic generalization, we used abstraction in context to analyse a classroom focus group and the mathematical progress of classroom community. Analysis allowed us to follow the evolution of construction of meanings for the concept of variable. It also indicated the critical role of the teacher and available tools in knowledge shifting when students generalise while modelling a realistic situation. We focus here on the students' knowledge transformations about the concept of variable through motion as they worked in groups or the whole class setting.

Keywords: patterns, generalisation, variable, abstraction, classroom

INTRODUCTION

Generalisation of patterns is considered a way to introduce algebra (Radford, 2018). The existing literature indicates that the use of a visual aid in presenting problems involving the search for patterns can lead to the application of different approaches to achieve generalisation, either of visual or nonvisual nature. However, teachers traditionally tend to mainly emphasise the numerical patterns and not the visual ones. In many situations, this can be problematic when trying to help students reach an algebraic expression which generates any term of the pattern (Barbosa & Vale 2015). Also, even though students can identify and predict patterns (Mason, 2005), they are unable to articulate a general relationship in natural language or in algebraic symbolism. Existing research around patterns has shown that this difficulty perhaps arises because patterns are presented mainly in a static way through images that show some of the early stages. Yet, students are asked to construct near and far generalisations for any stage or figure number in the pattern through questions such as: How many are here?; How many are there?; and 'How many are there in general?' (Rivera, 2007; Radford, 2018). Specific studies (e.g., Mavrikis et.al., 2013) have reported that students aged 12–13 usually resort to counting in order to answer 'How many' questions, especially if the thing being counted is static.

Another reason for this weakness could be most students' superficial understanding of the concept of variable that lacks multiple representations. Variable is the main component of algebraic notation and used to express mostly unknown quantities. Researchers (Mavrikis et.al., 2013) have indicated that the process of algebraic generalisation requires recognising a relationship between quantities and variables and expressing it through general statements. In recent research, teachers' predominant teaching approach centred on the treatment of 'letters as unknowns' rather than as variables. Additionally, teachers used algebraic patterns to support students' developing appreciation of an algebraic variable as a dynamic concept (Clark-Wilson & Hoyles, 2017). An approach through patterns, especially in a figural form, has been proposed (Rivera, 2007) as a fruitful way to introduce students to the notion of variable. Also, the combination of figural patterns and real-life contexts seem most suitable for supporting students' developing knowledge of variables and ability to express generalisations symbolically (Maj-Tatsis & Tatsis 2018).

As regards the use of digital tools in pattern generalisation, recent research approaches have suggested microworlds' reliability. This is because when a mathematical microworld is placed within a realistic context which is meaningful for the learners, it can build a basis for progressive algebraic generalisation (Mavrikis et.al., 2013). The realistic scenario microworld designers choose most often is that of motion in the sense of animation. This is because motion is not just an illustration but a paradigm for most of the part of mathematics that examines change. This evolution from static to dragging, sliders and animation has presented new challenges. These concern specifically the role of teacher and researcher claims that the dynamic motion context supports teachers in rethinking their own mathematical knowledge and developing more precise mathematical responses to the meaning of a variable provided by the dynamic mathematical technology (Clark-Wilson et. al. 2017). Also, applying an embodied perspective to the design of animations facilitates understanding of a dynamic system and be enhanced by asking learners to re-enact or follow movements through gesturing (de Koning & Tabbers, 2011). However, many teachers lack confidence in using dynamic affordances in their teaching. This creates a difficulty in studying interactions in the complex classroom environment with digital tools and the role of the teacher (Clark-Wilson et. al. 2020). For instance, there have been studies over the last decade based on eXpresser: a specially designed microworld with explorative tasks aimed at linking figural patterns to algebraic relations. The Geraniou & Mavrikis 2015 study indicated the need for further studying how the available software structures, the teacher and classroom community mediate students' generalisations as abstractions. Also, despite some successes, difficulties remain, and these tend to coalesce around the need for significant pedagogic support from the teacher to provide a bridge to algebraic symbolism, especially variable and generalisation.

Our literature review in the field revealed that over the past twenty years, there has been a plethora of studies and research findings on algebra that focused on algebraic thinking through patterns with or without the use of technology. Despite that, we still know very little about the evolution of meaning making for algebraic generalisation in the classroom environment. For this reason, in this study we investigate year 8 students' construction of meanings for algebraic generalisation. The students collaborated in groups using the exploratory microworld eXpresser to create the pattern of a bee hive by expressing it through repeated building blocks of square tiles and developing the rule underpinning the calculation of the number of tiles in the pattern. The microworld allows students to use iconic variables to reproduce their constructions for a different number of repetitions, to express generalisation and to check their correctness through appropriate feedback. In this paper, we focus on the role of teacher, classroom community and available tools to students, proceed to algebraic generalisation and then consolidate the concept of variable through motion.

THEORETICAL BACKGROUND

A theoretical framework for investigating processes of constructing abstract mathematical knowledge is Abstraction in Context (AiC) (Dreyfus et.al., 2015). AiC offers a way to describe how meanings are constructed by shedding light on their connections through three epistemic actions: recognising (R), building-with (B) and constructing (C). Recognising an already known mathematical concept occurs when a student recognises it as inherent in each mathematical situation. Building-with involves combining existing knowledge elements to solving a problem or justifying a solution. Constructing is carried out by assembling or integrating previous knowledge elements to produce a new structure. The model suggests constructing as the central epistemic action and constructing actions are at times nested within more complex constructing actions. Those constructs that are nested in others are called sub-constructs. This progressive, familiar epistemic action is called consolidation. It expresses the increasing ability to recognise and of building-with. The model is called the “RBC+C-model.” In it, the second C is used for Consolidation (Dreyfus et al., 2015). The RBC methodology begins with a priori analysis of the task, in terms of the knowledge elements that students should construct. Those knowledge elements are then used as milestones in the analysis of the processes of construction and the constructs that will emerge for the students during their engagement with the task.

According to Radford, learning occurs when the cultural knowledge has been transformed into knowledge for students, through students’ and teacher’ joint activity (Radford, 2022). In our attempt to contribute to the research about generalisation of patterns and to see what happens in the evolution of students’ meaning making for algebraic generalisation, we examine this joint activity that Radford called “joint labour”, as a path through the lens of AiC. To shed light on the transition process we consider the construction of knowledge as an abstraction process where construction corresponds more with the notion of knowledge transformation and less by knowledge construction. So, in short, we make a systematic investigation of the joint labour by analysing it with epistemic actions, and by identifying knowledge elements in algebraic generalisation of patterns. Influenced by the discussion of the transformation of knowledge through classroom interaction (Dreyfus et.al., 2015) and its contributions to current research on the role of the teacher, our paper focusses on how the teacher, classroom community and available tools contribute to the creation and synthesis of those processes so that students can reach algebraic generalisation and consolidate the concept of variable through motion.

EXPRESSER

eXpresser is a mathematical microworld designed to support 11–14 years-old students in constructing figural patterns using coloured square tiles. More precisely, includes a set of actions that can be carried out so that students can build patterns and test their generality through animation (Mavrikis et.al., 2013). Some key features of eXpresser are visible in Figure 1. It consists of two main areas: (a) My Model (a work area, on the right of the screen); and (b) Computer's Model (on the left). In My Model, students can use building blocks of square tiles to make patterns that can be combined to models. EXpresser allows students to work with icon variables so as to reproduce dynamically (i.e. to ‘animate’) their patterns for different numbers of repetitions, express generality through semi-algebraic relations, and test the validity of these relations for random values of repetition through appropriate feedback. By default, all numbers in eXpresser are constants and it is possible to change their values by “unlocking” them to become variables. Students can convert a constant number to a variable in My Model while in Computer’s Model, the variables take random values. Students can construct a model rule for the total number of tiles. If the pattern is correct, it will be coloured. Otherwise, as an indication of error, the pattern appears with no colours. Finally, when students find the general rule

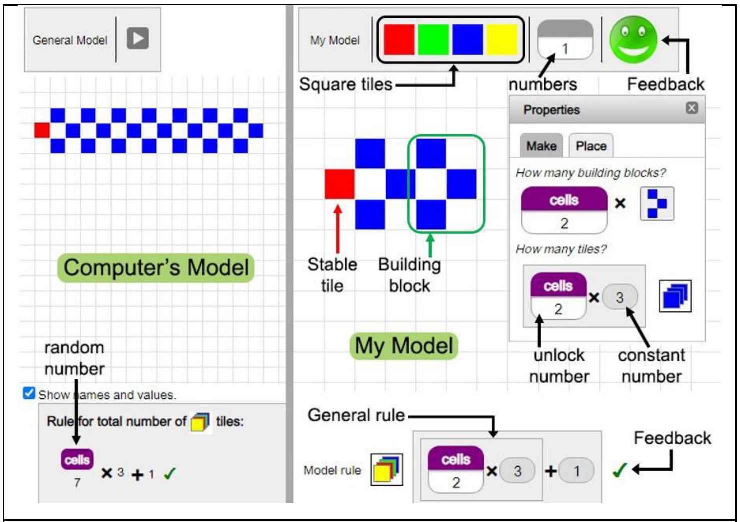

Figure 1. Pattern on eXpresser

in Model Rule, the sign " ✓ " appears and the face icon on the central toolbox turns green and "smiles". From our perspective, it is crucial to see how this microworld, which can provide motion (in the sense of animation) to patterns, the setting of a classroom in coordination to the realistic context of a hive coexist and shape knowledge transformations with which students can reach algebraic generalisation of patterns.

METHODOLOGY

The data analysed here come from a larger study in a lower secondary experimental school in Athens. Our research approach is informed by the idea of "design" in learning (Prediger et al., 2015) and task design aims to link patterns to a problem associated with the realistic context of a hive (Figure 2). Thus the need for algebraic generalisation was expected to arise as part of a beekeeper's problem solving. 24 Students (year 8) worked in six groups of four in 25, one-hour sessions over two months. Analysis in this paper focussed progressively on students' meanings making for the concept of variable through motion. The class teacher implemented the task and acted as facilitator, challenging the students with specific reference to the authentic context. A researcher was a participant observer who handled four cameras and six recording devices for data collection and specifically captured the focus group and the whole classroom experimentation. The data were fully transcribed for the analysis. Episodes were analysed through the lens of AiC. Classroom's utterances that appeared to be crucial in students' conceptualisation and expression of generalisations were coded through the RBC+C model. The task "Beekeepers" requires the exploration of a two-dimensional pattern consisting of several non-linear patterns. This task is designed to let the students experience patterns with two variables: the length and width of a honeycomb. The aim is to find the most economical pattern of waxes, so students are asked to compare triangle, square and hexagon in relation to the economy of perimeter and surface coverage. Students have to explore the building blocks through manipulatives, so the teacher gave them cardboard squares and tire-up sticks (Figures 3, 4) to recognise the pattern's building block. Then students construct those patterns in eXpresser and find an algebraic formula to describe the total number of honeycomb walls. The expected formula is 2xy+3x+y+1, where x is the number of cells in a row and y the number of series of cells vertically.

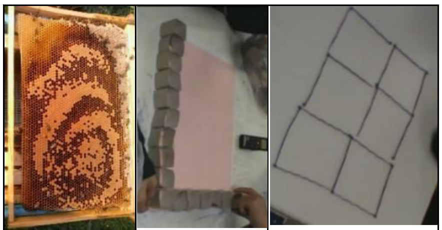

Figure 2 **Figure 3** **Figure 4**

A PRIORI ANALYSIS OF THE TASK

From a priori analysis, the main knowledge elements in this task were: the construction of knowledge elements C_{bb} – building block, C_{un} – unlock number, C_{mr} – model rule, where sub-constructs were nested in C_{un}: C_{un2} – 2nd unlock number, C_{mo} – motion and in C_{bb}: C_{bb2} – 2nd building block. The main difficulty of this task is that the hive pattern consists of two different non-linear patterns.

RESULTS

Our results came from one small group's work and a whole class discussion in our attempt to show how the concept of the variable transformed through students' and teacher' "joint labour". We analyse focus group's work on the hive pattern (Episode 1) and close our analysis by a whole class discussion. In the latter, the teacher highlights the motion that two separated variables provide, and students consolidate the concept of variable (Episode 2). Episode 1 took place during the 17th teaching hour where while working in eXpresser, students recognise the need for the 2nd building block (C_{bb2}) for two different patterns series of squares (Figure 5). As students' arguments still seem weak, during the 18th teaching hour the teacher called for a whole class discussion by inviting the students in the focus group to share their findings with the whole class. All names used in the dialogue of the two episodes are pseudonyms.

Episode 1: Recognise the need for two variables.

Although students subsequently construct the concept of two independent building blocks (C_{bb2}), they have not yet recognised the need of two independent variables (C_{un2}). For this reason, the teacher, (1) tries to mention the need of independent number of columns and series in the honeycomb, as they work in groups.

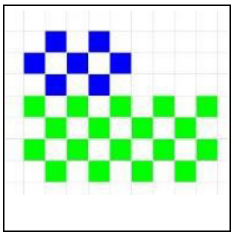

Figure 5

1	Teacher:	Could you show me a hive with 10 columns and 3 series?
2	Chloe:	No, we have "walls" [unlock] everywhere, so all change simultaneously.
3	Teacher:	But we want a different number of series and columns [show to the right/down]
4	Kate:	So, we don't want the "walls" everywhere, lets create another one, unlock number named "toi"!
5	Teacher:	Find a name that makes more sense. What does it mean "toi"?
6	Chloe:	Just a way to say walls. So, unlock number is the number of cells in the first series, let's name it cells! ***[Connection Cun2– Cna])***
7	Teacher:	And now what is at 3? [The teacher changes unlock number from 4 to 3.]
8	Kate:	Its series… it's the number of the series that consists of 2 walls each time, so, let's name it "series". ***[Connection Cun2– Cna]***

Here, students have to deal with handling two separated unlock numbers (C_{un2}) and their names, something they (Chloe 2) seem to recognise (R/C_{un2}). After the teacher's interventions (Teacher 1, 3), they (Kate 4) also demonstrate building with (B/C_{na2} - C_{un2}) through naming. Next, the teacher helps students construct the 2nd unlock number (C/C_{un2}) by requesting students to give meaning to the name (Teacher 3). Also, the teacher indicates for the first time how to manage two different directions (Teacher 3): one for the columns and one for the series. The teacher pays attention to the motion of the pattern pointing right and down with their hands (R/C_{mo}). Students through name negotiation, came up with two names, cells for the first unlock number (Chloe 6) that controls the number of cells in each series, and series for the second (Kate 8) that controls the number of the series. In this episode we see how students cooperate with their teacher to approach the creation of a hive with different width and length. In short, the group developed a pattern of square cells with only one unlock number (C_{un}) that simultaneously controls the rows and columns (Chloe 2). Through the discussion with their teacher students recognise that they need more unlock numbers (C_{un2}) that at the same time means different things (Chloe 6, Kate 8). It is worth mentioning that as students are building with (B/C_{na2}) − the name of the 2nd unlock number − they then construct (C/C_{un2}) − the meaning of the 2nd unlock number −(Chloe 6). When they construct the name of 2nd unlock number (C/C_{na2}), they then consolidate the meaning of 2nd unlock number (C/C_{un2}).

Episode 2: Consolidate two separated variables that moves.

In episode 2, students consolidate the concept of variable through the motion that eXpresser provides and discuss in the whole class the need for two separated variables. This is an episode that shows us the crucial role whole-class discussion plays in consolidating the concept of variable.

9	Teacher:	Unlock number determines how many building blocks I use. Give it a name!

10	Students:	Cells.
11	Teacher:	Cells well, and there, in model rule, what should we fill in there??
12	Kate:	Cells multiply by 3, plus one for the front square to close the first cell.
13	Teacher:	Ok, now the second pattern of the series. Would this be the same pattern?
14	Mia:	No, there are already walls above so we need 2 extra walls each time. Put 2 lime [-coloured] squares for the new building block and unlock the number.
15	Teacher:	Yes, but now we have 4 cells above and 5 below. How grow them together?
16	Kate:	We have to replace unlock number of the second series with the “cells”.
17	Teacher:	And what do we have to insert in the model rule now?
18	Mia:	Cells multiply by 3 plus 1, plus cells multiply by 2 plus 1.
19	Teacher:	Ok, but we have only two series. How can we create more downwards?
20	Kate:	Make a new pattern with the 2^{nd} series, unlock a number and name it “series”!
21	Student:	No, “columns”!!!
22	Teacher:	The columns grow vertically [moves hand to the right] and series horizontally [moves hand downwards]. What change do we make now in model rule so all series are coloured?
23	Chloe:	“Cells” multiplied by “series” multiplied by 2 because each cell has 2 walls.
24	Kate:	So, cells used to move horizontally, while series move vertically!
25	Chloe:	Move them to check. See, when you change “cells,” the blue pattern moves right together with the lime [-coloured] one, and when you change “series,” the lime [-coloured] pattern moves down.
26	Mia:	Unlock number series is down and cells is right! ***(Connection Cmo– Cv)***
27	Teacher:	So, can you describe the total number of walls with a formula?
28	Kate:	2 *series*cells + 3*cells +series +1, where 2*series*cells refers to the whole lime part of the pattern, 3*cells to the first row of blue squares, series to the column of lime squares that closes the lime part of cells and the 1 for the square that closes the blue row of cells. Series is the number of rows and cells is the number of columns. ***(Connection Cmr– Cbb2– Cun2–Cmo–Cv).***

In this episode, two noteworthy ideas are developing. First, that unlock number describes the number of building blocks that are the cells. Recall that in episode 1 (small group work), this idea was put forth as a claim. Now in the whole class, it appears as data. More precisely, it is interesting that unlock number is renamed (B/C_{na2}) from “walls” in the previous group work to “cells” in the whole class discussion (Kate 12). Moreover, in this episode we see that many knowledge elements found in the group work were on the building with phase. Now in the whole class discussion, the knowledge elements are constructed or consolidated through the presentation. For example, students consolidate (C/C_{bb2}) the 2^{nd} building block (Mia 14) and construct (C/C_v) − the concept of variable − using the same unlock number in order to create the same amount of cells in the two different patterns of the 1st and 2nd series (Kate 16). The second noteworthy idea developing here is the merging of the two interpretations for unlock number “cells”, namely that of the number of cells in a row and the interpretation of unlock number as a tool which gives motion to the right (Chloe 25). Here we also have corresponding interpretations about unlock number “series” that controls the number of series to the motion down (Mia 26). Regarding the RBC model, the teacher once again calls attention to the motion with by making corresponding hand gestures. This helps students recognise (R/C_{mo}) this extra feature of unlock number (teacher 22). Students demonstrate building with (B/C_{mo}) this new feature (Kate 24) by associating each variable with a separate move and construct (C/C_{mo}) it (Chloe 25) by confirming moves in software by manipulating variable sliders. In addition, Mia (26) consolidates it (C/C_{mo}) by describing the connection of unlock number with motion in reverse when she says 'Unlock number "series" is down and "cells" is right'. Last but not least, we see the gradual building with (B/C_{mr}) of model rule (Kate 12, Mia 18), then its construction (C/C_{mr}) (Chloe 23) and finally, its (Kate 28) consolidation (C/C_{mr}). As a result, the students finally achieve representing a semi-symbolic formula. This is possible because, the concepts of 2^{nd} building block, 2^{nd} unlock number and concept of variable had been consolidated.

CONCLUSIONS

As shown in our analysis, by using the RBC model, we found that constructs corresponded to the knowledge element of variable. For example, we found that the unlock number controls the rate of change that features a variable with motion. The key element was the "unlock number" of eXpresser. The concept of the variable (C_v) seems to be built-with and be constructed through the manipulation of the unlock number (C_{un}) and linking it with the other software elements simultaneously via the different task features, such as the number of cells and rows. Furthermore, students tried to express their thoughts through contextual task elements and build links between prior knowledge constructs. For example, we observed an evolution in the way of giving name to unlock number (C_{na}). At first, the names were given by chance, but then evolved to be more related to the task activity. In the task, the main difficulty was that students had to manipulate two independent unlock variables (Cun2) with independent names (Cna2). This added another feature to the concept of the variable (C_v), the property of motion (C_{mo}), that students conceptualised as the way to move to different directions, to the right and down. Finally, the consolidation of the concept of variable came about because of an attempt to connect all the previous consolidations of nested sub-constructs that are motion, 2nd unlock number, 2nd name, and 2nd building block to create the model rule (C_{mr}).

The teacher contributed by going back and forth between small group work and whole-class discussion and highlighted crucial knowledge elements when it was necessary. These included the need for different building blocks, motion through unlock numbers and names of variables related to context of the hive. The motion context, supported by the animation, enabled the teacher to make the important connections between knowledge elements. Moreover, the teacher indicated the need for more than one unlock number, focused the students' attention on finding relationships between the unlock numbers, and find meaning in each software element. It's worth mentioning that within the group work, we observed that after the teacher commented, the result was: students tended to connect some knowledge elements, recognise others' knowledge elements, and/or push meaning making of those elements higher. The analysis provides evidence of the importance of whole-class discussion, which was an occasion for participants to develop ideas beyond what had been constructed in small groups. More precisely, as students tried to share their ideas with the whole class, ideas gradually be condensed through discussion, as for example we observed in the students' attempt to present the model rule. Additionally, many knowledge elements present in the group work were in the building-with phase. Then in the whole class, these knowledge elements became constructed or consolidated through the presentation. We observed this, for example, in the students' attempt to describe what happens in the pattern where they reached to consolidate the model rule. As this paper has shown, the diversity of available tools and the classroom community are bearers of great potentiality. They can make the learning processes, that mediate until generalisation becomes visible, emerge. We saw that the construction of the meaning of the concept of variable initiated through different collaborative and individual social settings, including the classroom where students moved between working alone, group work and whole-class discussion and technological tools supporting the process of abstraction. In summary, we saw that algebraic activity is characterised by actions of representing and manipulating and because of its abstract character, algebra is not an easy subject to teach without any context. Although, in this paper we have provided some evidence to the evolution of student's meaning making for the concept of variable through motion, both the critical role of the teacher and available tools in knowledge shifting when students generalise while modelling a realistic situation still deserves more investigation.

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