OBSERVING THE SPACE THROUGH THE PLANE: ANALOGIES PROMOTED WITH DYNAMIC GEOMETRY ENVIRONMENTS

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Research in mathematics education on the use and benefits of dynamic geometry environments is notable in 2D geometry, but its development is not similar in 3D geometry. Although some proposals have been formulated to explore 3D geometry, in which dynamic geometry environments are used and the analogy between objects and relationships between the domains of 2D and 3D geometry are involved, there are aspects about the use of these environments in 3D geometry that need to be studied, given the differences in the representation and manipulation of geometric objects. In this document we show the analogical reasoning displayed by a mathematically gifted student when solving construction-and-proof problems in a 3D dynamic geometry environment, the way this is promoted by some limitations imposed by the software, and some problems the student faced in this process.

Keywords: Analogical reasoning, three-dimensional geometry, dynamic geometry environment, construction-and-proof problems

INTRODUCTION

Nowadays there is abundant specialized literature that shows the benefits and drawbacks of integrating dynamic geometry environments (hereafter, DGE) in the teaching and learning of geometry (Sinclair et al., 2016). Although research has made great steps in 2D geometry, the advances research has made in 3D geometry is not similar (Gutiérrez & Jaime, 2015). Intuitively, it could be considered that the research results involving 2D-DGE could be extended to configurations in which 3D-DGE is used. However, practice shows that some characteristics of the latter, such as the way of representation and manipulation of objects on the screen, make the relationship between an individual and the 3D-DGE be quite different from what happens when using 2D-DGE. It configures a terrain with particular characteristics that deserves to be studied.

3D geometry is not frequently studied in schools, due to the difficulty of making plane representations of 3D geometric objects capturing all their relevant properties or allowing make a correct reading of the properties that characterize the embodied object (Parzysz, 1988). As 3D-DGE offer a possibility to interact with 3D geometry objects (Sinclair et al., 2016), some proposals for the study of 3D geometry have involved this digital resource, as well as analogies between objects and relationships of 2D and 3D geometry (Ferrarello et al., 2020). This kind of approaches to teaching 3D geometry may allow students discover some properties preserved in both 2D and 3D geometries and understand the reasons for other properties to be valid in 2D geometry but not in 3D geometry, or vice versa.

As part of a research project in which we analyze the processes of reasoning and the progress in learning to do deductive proofs by several mathematically gifted students when solving construction-and-proof problems [1] in a combined 2D and 3D-DGE, the research objective of this document is to analyze the analogical reasoning exhibited by a student when solving the problems, to show how this reasoning is restricted by the lack of some tools in the 3D-DGE that he had used in the 2D-DGE. The analysis also alerts on the difficulties faced by the student when trying to extend some properties from 2D geometry to 3D geometry and the role played by some 3D-DGE functionalities in this process.

THEORETICAL BACKGROUND

What is analogy

According to (Schlimm, 2008, p. 178) an *analogy* is "a relation of similarity between two *domains*, where a domain is a fixed representations of certain aspects of a phenomenon, situation, process, model, problem, conceptual structure, etc.", through objects and specific relationships. By establishing a set of relationships in one domain and taking them to another domain, in which these relationships are valid, analogy allows the formulation of hypotheses and the simplification of complex mental operations in the second domain, when these are performed in the first domain, that is already known (Fishbein, 2002).

Making analogies is also a product of human activity, with several benefits (Richland & Simms, 2015). This process demands recognizing corresponding conceptual structures between different domains and their similarities or differences, thus advancing from a comparison based on superficial aspects or characteristics to one based on relationships between those domains. It means that this process has implications for the learning of mathematics (English, 1997), since new objects and relationships in an unknown domain can be discovered by extending ideas from a known domain.

In our study, analogy becomes a tool that allows students explore and discover relationships in the unknown (for them) domains of spatial geometry, starting from known domains of plane geometry. Likewise, the analogy offers the possibility of solving problems in a domain of spatial geometry, when they are formulated and solved in the corresponding domain of plane geometry, so that the results obtained in plane geometry can be interpreted in light of spatial geometry.

Analogical reasoning: establishing and using of analogies

Analogical reasoning is the process through which information from different domains is represented as relationships systems for study and comparison according to certain objectives (Richland & Simms, 2015). Students' mathematical problem-solving activity has been analyzed to recognize the nature of the analogical reasoning exhibited by them. Although there are different processes with a specific role in this scenario, major emphasis has been placed by researchers on five of these processes: representation, retrieval, mapping, adaptation, and learning (Novick, 1988). These processes are described below according to our research interest.

At the beginning of a problem-solving process, in our case situated in 2D and/or 3D geometry, the problem is *represented* to identify the domain knowledge involved and the ways the elements provided by the statement of the problem may be related. With this representation, the solver can *retrieve* representations of knowledge or problematic situations that are relevant to solve the problem, which belong to 2D geometry. Having identified a domain in 2D geometry that provides information to determine the solution to a problem located in a related domain of 3D geometry, leads to the need of *mapping* the 2D and 3D domains, i.e., to establish links between objects and relationships in the 2D domain that will be useful with objects and relationships of the 3D domain. This does not imply that the relationships coming from the 2D domain are transparent to solve the current problem, since they are in a specific configuration, which may not be near the 3D configuration that is being addressed. This leads to the need to *adapt* the retrieved information (objects, relationships, procedures), so that it can be used in the 3D domain that is now being explored. Finally, when a problem in a 3D domain is solved, the strategy used, the information retrieved, and the relationships established between the 2D and 3D domains lead to new *learning* in which these elements intervene. Across this learning, there is an acquisition of new knowledge, as well as a possible modification of the previous conceptual representations of the solver.

All the analogical reasoning processes that have been described lead us to suppose that analogical reasoning is present in situations in which the representation and analysis of two domains leads to the establishment of analogies or to use an already existing analogy between two domains to solve problems or broaden existing relationships between those domains. The route that these processes outline is not easy to follow; English (1997) mentions the complexity that novice students face when carrying out this activity. For English, students require a cognitive maturity that allows them to focus on structural aspects of the compared domains and not only on contextual and superficial elements. This cognitive maturity is a characteristic differentiating mathematically gifted students from the average students in their grade or age.

METHODOLOGICAL ASPECTS

We present a case study, drawn from a broader research project in which we analyze the learning of proof, in the context of 3D geometry with the mediation of GeoGebra, by four Spanish mathematically gifted students. The students were 11 to 14 years old and studied in grades 1 to 4 of secondary school. We recognize them as mathematically gifted because, besides the ordinary schooling, the students had participated in programs of attention to general giftedness (AVAST) and mathematical giftedness (ESTALMAT).

We designed and implemented a sequence of 18 construction-and-proof problems in 60-minute sessions. Some problems requested the construction, first in 2D and then in 3D, of a geometric object satisfying some properties associated with equidistance (e.g., an equilateral triangle given its side). Other problems requested the construction of a 2D object and an analogous 3D one (e.g., the center of a circle in 2D and a sphere in 3D). The process of solution of each problem provided students with useful instrumental and conceptual elements to solve subsequent problems. For each problem, the students first had to solve it and then to discuss the solution and justify its correctness with the teacher (the first author of the paper), who led the conversation. The sessions were audio and video recorded after informed approval by students and parents. As students were in different school grades and had different previous knowledge, the teaching sessions were organized as individual clinical interviews.

We present episodes of the solutions of three non-consecutive problems by one of the students, named Hector (pseudonym). We have chosen these problems because they illustrate the elaboration and use of analogies between domains of plane and spatial geometry, as well as the influence of GeoGebra in this process. To analyze the analogical reasoning exhibited by Hector in solving the selected problems, we consider the five processes of analogy presented in the previous section.

THE SUBSTITUTION OF THE CIRCLE BY THE SPHERE

Prior to the implementation of the sequence, Hector's knowledge of spatial geometry was scarce, it came from his school experience and was limited to recognizing the sphere, some polyhedra and other simple solids. Hector's experience with GeoGebra was limited to the use of some 2D tools; GeoGebra 3D and its tools were unknown by him. The problems that precede the first problem we presented gave Hector a first contact with GeoGebra 3D, its *Parallel Line* and *Perpendicular Line* tools, and point and perspective draggings [2].

Problem 04: An analogy between circles and spheres

The objective of this problem was to introduce the circle and sphere as loci and suggest geometrical relationships between them. The first part of the problem, to be solved in GeoGebra 2D, asked to create a point at the same distance from three given non-linear points. Solving this part of the problem led Hector to involve the definition of circle as a locus, known by him from his school experience, as support for the properties that the requested point should satisfy.

The second part of the problem, to be solved in GeoGebra 3D, presented a red point A and black points B, C, D, and E (Figure 1a). Hector had to drag points B, C, and E so that the distance between them and A was equal to the distance between A and D. After building the segments determined by A and each black point, as well as measuring their lengths (Figure 1b), Hector looked over the tools associated with circles, although he did not find useful any of them. When the teacher asked him about the aim of this search, he expressed *the need for a circle with two points, the center, and the point through which it passes* ..., which could not be built because GeoGebra 3D does not have a *Circle (Center - Point)* tool. The conversation with the professor led Hector to understand that this tool was not available because it is possible to create several circles in space with the same center and containing the other point: *I already know why it is. It is because the circle can be this way* [outlining a circle with the pointer]. *The circumference can* [also] *be like this* [changing the perspective and sketching a second circle] ... *it can be in many ways, rotating*.

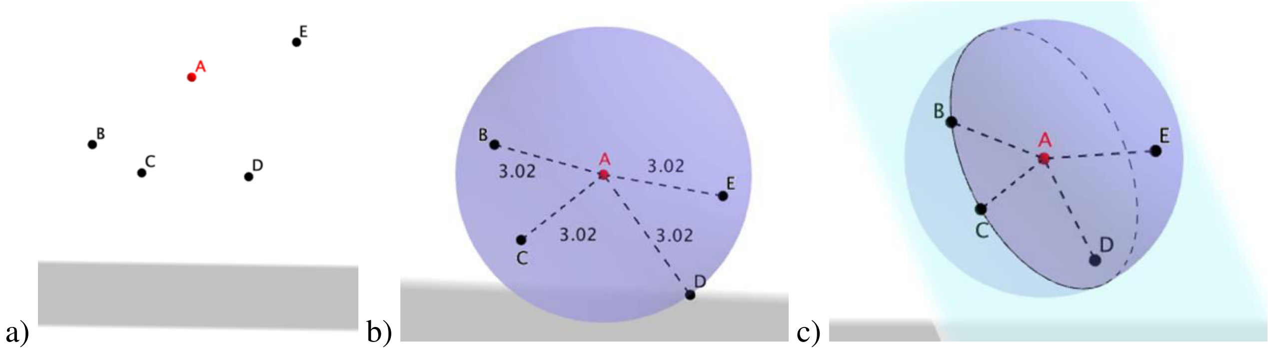

Figure 1. Substituting circles by spheres

Hector changed his strategy, he selected the *Sphere (Center - Point)* tool and constructed the sphere with center at A that contains point D. Points B, C and E were then linked to the sphere with the *Attach/Detach Point* tool (Figure 1b). Hector justified his actions by saying that *in the end it's the same. I just have to make* [the points] *fit exactly there* [on the sphere]. He explained his construction assuring that *A is undoubtedly the center, because I have used it as such. So, just as it happened with the circle, there is a property that is that A is at the same distance from any point that is on the circle, that is, on the surface of the sphere.*

Until now, the circle and the sphere had not been discussed, so the teacher asked Hector to express what each of these objects meant to him. Hector said that the sphere is a solid in which *the center is equidistant from all the points*, while the circle is *a geometric shape… the center is equidistant from all the points*. In the end, in what would seem to be a way of seeing the sphere as a general object that encompasses the cases of the circles that he previously pointed out, Hector mentioned that *in the circle that is in the plane shared by B and C* [and A], *A is in the center* [Figure 1c], *so* [points B and C] *have the same distance from A. On the circle that is in the plane of B and E, A is in the center, so they are equidistant from A.* [On the circle that is] *In the plane that E and C* [and A] *or D* *and C* [and A] *can share* [plane determinated by these points]*, well, the same thing* [A is in the center] ...

Problem 07: Using an analogy to construct a triangle in 3D

The objective of this problem was to construct an equilateral triangle given one of its sides in GeoGebra 2D (segment GH, Figure 2a) and later in 3D (segment AB, Figure 2b). The following episode shows the use of the analogy between circle and sphere that was elaborated in problem 04. In the 2D part of the problem, Hector built circles with center in G and H and radius GH. Then, the point I, intersection of the two circles, allowed him to build the requested triangle (Figure 2a). In the 3D part, although Hector looked at the available circle tools, he selected the *Sphere (Center-Point)* tool and built spheres with center in A and B and radius AB (Figure 2b). Hector created a point D at the intersection of the spheres and the triangle ABD.

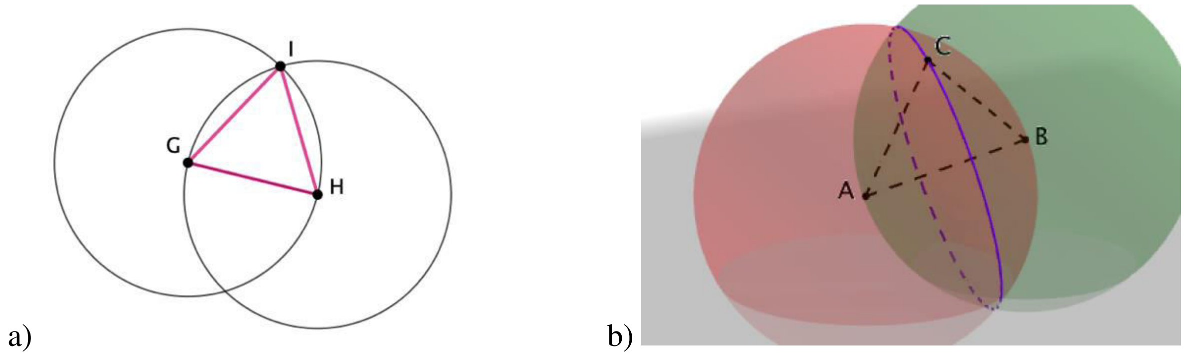

Figure 2. Construction of an equilateral triangle in 2D and 3D

Hector justified the validity of his construction by using the analogy circle-sphere to prove the congruence between the sides of the triangle: *I have done the same as in the 2D version, only that instead of circles I have used spheres... I have used spheres of radius AB, so these two segments here* [pointing to AD and DB] *are radius of circles with radius AB. So, they are all the same.*

Problem 16: A problem using the analogy

The last episode we present is the solution to problem 16, which asked to construct the center of a given sphere (Figure 3a). Hector made two unsuccessful attempts to use analogy to solve the problem, based on procedures that Hector had used to solve other problems. The first attempt corresponded to a construction that allowed him to obtain the center of a circle in the plane with the support of the tangent lines from two external points and the bisectors of the angles determined by these tangents (Figure 3b). The second unsuccessful attempt was based on the property that any angle inscribed in a semicircle is a right angle (Figure 3c).

a) b) c)

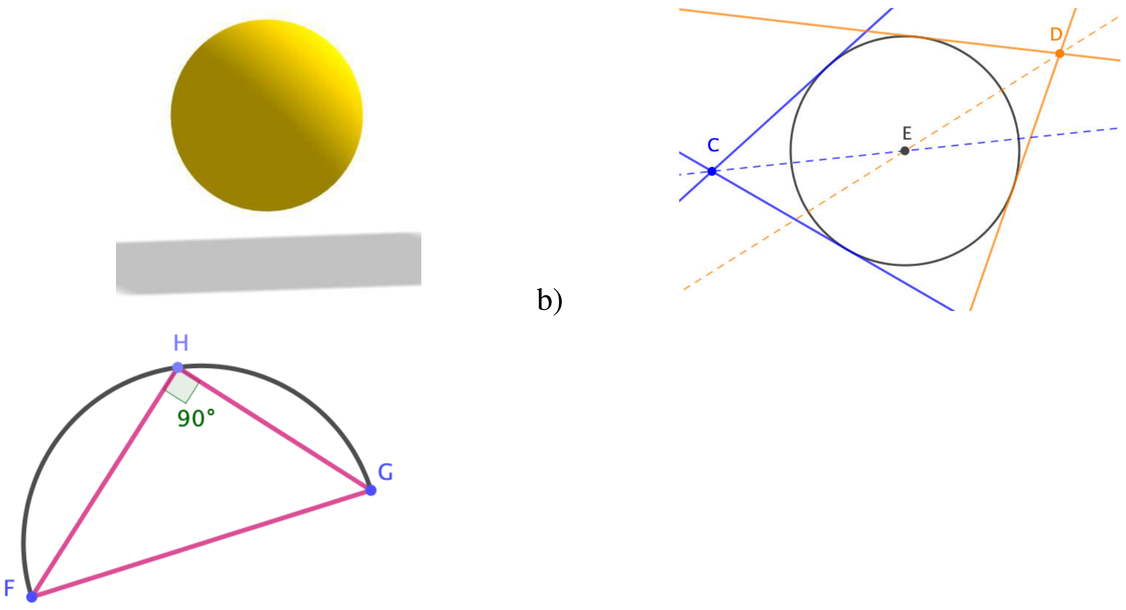

Figure 3. Two successful strategies in 2D

In his first attempt to obtain the center of the sphere, Hector constructed a point external to it, selected the *Tangent* tool, and clicked on the point and the sphere several times, without any result. His intention to recreate the construction he had made in 2D was not successful and, as he recognized, the reason for his procedure not working was that now there was a sphere: *I choose a point and then a circle... ah! It cannot be done with a sphere*.

In his second attempt, Hector constructed points E, I, and D on the sphere and point H as the midpoint of E and D; then he determined the measure of angle EID (Figure 4a). Without changing the viewing perspective of the construction, Hector dragged point I until the measure of angle α was close to 90º (Figure 4a) while stating that *If I get point I* [so that] *angle I to measure 90º, H will be the center of the circle*. However, the teacher asked him to change the viewing perspective of the construction, with which he noted that this strategy was not useful either (Figure 4b).

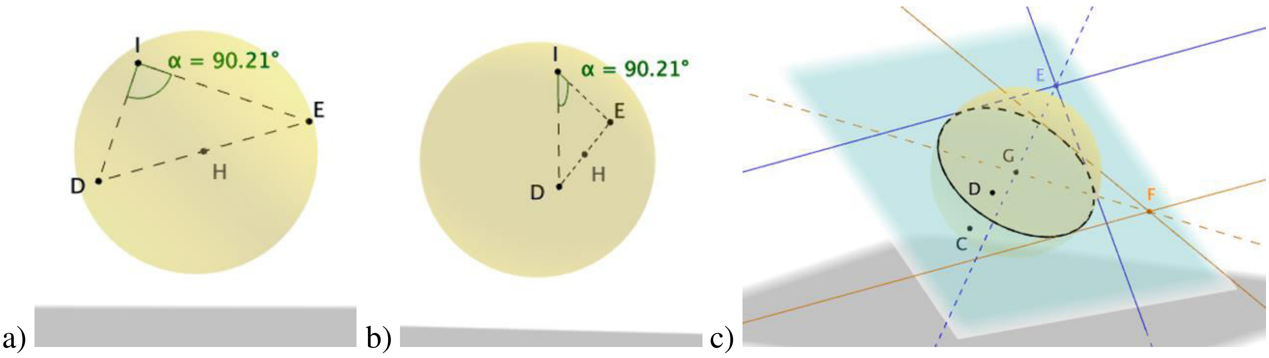

Figure 4. Modifying unsuccessful strategies in 3D

Hector expressed his aim to have a circle contained in the sphere: *the largest, that is, the Equator* [since this circle would have the same center as the sphere] ... *If I find the center* [of the circle], *I will find the center of the sphere*. To get a circle, Hector built the bisector plane of two points C and D in the sphere and the circle intersection of the sphere and the plane. He justified the correctness of the construction: *I knew that the center was going to be ... in the bisector plane of those points that*

I have made, because as the center [of the sphere] *is equidistant* [of the points in the sphere]*, it must be in that bisector plane*. Finally, Hector recreated on the bisector plane the construction he had made in 2D to determine the center of the circle and the sphere (Figure 4c). Hector justified the validity of his construction by stating: *I have treated it as if it were in 2D. I made a procedure that I also used in 2D ... When making the tangents, the bisector line of* [the angle between] *the tangents always pass through the center, that creates a diameter. And if I repeat it twice, it creates another diameter... the intersection of those two diameters gives the center of that circle, which is the largest and is like the Equator, has also been applied to the sphere.*

ANALYSIS

The episodes that we have presented show the elaboration and use of analogical reasoning, involving circles and spheres, as a tool to solve construction-and-proof problems in 3D geometry. This reasoning had its origin and development in the fact that some GeoGebra 2D tools are not available in its 3D configuration, as well as in the student's intention of taking advantage of his knowledge of plane geometry relationships to use them in space geometry, to simplify the solutions of the problems.

The episode from problem 04 shows the establishment of the analogy between the circle and the sphere. Although Hector knew well what he had to do and he contemplated a strategy to obtain the desired result (representation), not having the *Circle (Center-Point)* tool in 3D prompted him to provide an explanation for this fact and to introduce the sphere (recovery), arguing that the result obtained with circles or spheres would be identical, which was based on the definitions that he knew of these geometrical objects (mapping). Hector transformed the problem into an equivalent one when he considered the sphere and its definition to obtain the desired result (adaptation). The interpretation of the sphere that Hector used (the sphere contains every circle with center A and containing each given point), provided evidence of the generality attributed to the 3D construction, distinguishing it from the 2D construction, which would be a particular case that occurs in the plane (learning).

When solving problem 07, Hector made use of the analogy to build the equilateral triangle in 3D. The experience of building this triangle in 2D (recovery) gave Hector elements about the properties that should be satisfied (representation) and how these could be obtained in 3D with the help of spheres and the procedure used in the plane (mapping). Due to the nature of the problem, the need to adapt the procedures involved was not perceived, as this was almost immediate. In the end, the validity of the construction carried out in 2D with circles and the analogy between these and the spheres were used by Hector to support the validity of the result he obtained in 3D. Thus, Hector used the analogy for the establishment of results in a domain (3D) when making the comparison with what he knew in another domain (2D) (learning).

However, the last episode (problem 16) shows the difficulty that can be faced when using analogies and the inappropriate results that can be obtained. To construct the center of the sphere, Hector used two 2D strategies to determine the center of a circle (recovery). However, the way in which these strategies were incorporated into 3D did not yield the expected result, because adequate links between the relationships in 2D and the characteristics of the 3D objects available (mapping) were not established; this difficulty was evidenced when Hector used the *Tangent* tool, or when 3D objects were manipulated from a single viewing perspective on the screen, as seen when Hector tried to construct a right triangle.

DISCUSSION

Based on the comments above, we have showed that trying to solve construction-and-proof problems in 3D-DGE can evoke building strategies known in the 2D-DGE. However, not having the same construction tools in the 2D-DGE and 3D-DGE or having these tools different performance in both contexts, can become an opportunity to understand mathematically why this happens and to promote processes of analogical reasoning that help modify and amplify the construction strategies known in 2D to make them functional in 3D. Hector's solutions showed that analogical reasoning may not require a deep analysis of the domains involved. On the other hand, it is possible that mapping and adaptation processes are deep and necessary, and not take them into account or not carried out properly would provide different results from those expected.

The latter idea is related to the treatment of the representations of 3D objects provided by the 3D-DGE on the screen. The actions carried out by Hector when trying to construct the center of the sphere cannot be generalized, but we can conclude that extending known procedures from 2D to 3D is not straightforward and can lead to a simplification of the properties of the 3D objects involved by relying on a single screen representation of them, which may originate incorrect constructions. This may be due to an inadequate process of mapping or adaptation of the domains involved. However, some 3D-DGE features that allow the manipulation of objects on the screen can help to reduce this problem, an aspect that highlights features of the 3D-DGE such as perspective dragging.

The results presented come from the analysis of the particular case of a mathematically gifted student and are not intended to be generalized. However, they offer evidence of the nature of analogical reasoning when solving construction-and-proof problems in a 2D and 3D-DGE and how such kind of reasoning is favored or hindered by limitations imposed by the software, such as the unavailability of some construction tools. The results also offer elements in favor of the need to study more deeply the influence and risks of using a 3D-DGE in the exploration of 3D objects, given the differences in the representation and manipulation of geometric objects in 2D and 3D.

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NOTES

1. In a DGE, construction-and-proof problems ask i) to create on the DGE a geometric figure having some properties required by the problem, that must be preserved under dragging, and ii) to prove that the procedure used to create the figure is mathematically correct, by explaining and validating the way of construction (Mariotti, 2019).

2. *Perspective dragging* is a term used by Leung and Or (2009) to refer to the possibility to change the viewing perspective of 3D objects on screen when a three-dimensional dynamic geometry environment (Cabri 3D, GeoGebra 3D) is used.

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