TINKERING WITH ANIMATED MODELS OF LETTERS: INSIGHTS IN THE USE OF VARIABLE

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In this paper we discuss the generation of 9th grade-students' meanings of the concept of variable as a generalised number and their competence in using mathematical properties while tinkering with animated models of letters of the Greek Alphabet with the programming tool MaLT2. Using a combination of the 'Teaching for Robust Understanding' Framework and the '5Es' Framework in designing innovative activities, the study explores students' mathematical experience by focusing on how students use the concept of the variable in the mathematical properties embedded in the dynamic models of letters, and provides some insights into their misconceptions about the role of the variable as a generalised number.

Keywords: Mathematics, variable, programming, debugging, animated models.

INTRODUCTION

In this study the author investigates the meanings that students generate about the concept of the variable while tinkering with animated models of letters of the Greek alphabet and negotiating the use of mathematical properties embedded in the letter models in the programming environment of the digital tool MaLT2. The research doesn't aim to study how students learn a new mathematical concept or property, but how they put into use a concept or property in a context based on mathematical activity with generalised models through programming and engineering (Kynigos & Diamantidis, 2022). The variable plays a fundamental role in mathematics, but when it comes to its use as a generalised number, research shows a number of difficulties or misunderstandings for students in both algebra and geometry. The problems stem from the lack of the essential basis of the definition and functional role of the variable, leading to students' inability either to abstract and generalise or to deal with variables in other contexts of mathematics beyond algebra (Dede, 2004). At the same time, in geometry, students are reported to struggle with the Pythagorean Theorem or trigonometric numbers, when the use of variables is required (Rudi et al., 2020; Orhun, 2004). Some episodes are presented specifically about the use of the variable, which can have multiple meanings in mathematics, while both the tool and the designed activity are based on the use of the variable as a generalised number in geometry. The design of the activity, which involves the construction of dynamic models of letters using the programming tool MaLT2, was based on the approach that refers to mathematics as "ideas" rather than fixed "products", as usually found in school curriculum (Li & Schoenfeld, 2019) and considers knowledge as an active construction by the learner through the exploration and the expression that the digital medium offers. The capital letters of the alphabet and the programming environment have also been used in the past to explore students' meaning making around the concept of ratio and proportion (Noss & Hoyles, 1996; Psycharis & Kynigos, 2009). However, in this study, the researcher paid attention to different mathematical properties that could be embedded in the selected letters from the secondary school (variable as generalised number, Pythagorean theorem, trigonometry), without separating the content knowledge into different chapters and classes. Beyond this aim, the author made an attempt to deploy the Teaching for Robust Understanding (TRU) framework (Schoenfeld & the TRU Project, 2016) with the use of digital technologies, by using it in combination with the pedagogical framework of '5Es' (Benton et al., 2016; 2017) in designing innovative learning activities based on programming, and analysing students' mathematical experience during their interaction with the programming tool from an extended perspective.

THEORETICAL FRAMING

The Teaching for Robust Understanding (TRU) Framework

The theoretical framework of Teaching for Robust Understanding (TRU) provides a way of identifying powerful learning environments by shifting attention from the lesson content to the creation of experiences for students to be developed into powerful thinkers and problem solvers in class (Schoenfeld & the TRU Project, 2016). According to this, the quality of a learning environment depends on the extent to which it provides opportunities for students in the following five dimensions: *Dim.1: The Content (The Mathematics)* refers to the quality of students' mathematical experiences and the opportunities provided for them to make connections with previous concepts or procedures and to develop strong mathematical thinking, *Dim.2: Cognitive Demand* consists of the challenges ('productive struggle') for students to be actively engaged in mathematics and to make sense of important disciplinary ideas and their use, *Dim.3: Access* involves the meaningful engagement and participation of all students in the learning process, *Dim.4: Agency, Ownership & Identity* concerns students' willingness to get involved, ownership (feeling of control) of content and acquisition of a mathematical identity as people who can do mathematics through opportunities to generate and share ideas or arguments and to build on others' ideas and vice versa, *Dim.5: Formative Assessment* relates to the responsiveness of the environment to students' thinking. It is clear that the five dimensions are interrelated, rather than isolated, and according to the TRU framework, all dimensions must be met for a learning environment to be considered powerful.

The '5Es' Framework

The pedagogical framework of '5Es' (Benton et al., 2016; 2017) offers a way of thinking about and designing activities with programming media, by incorporating a constructionist approach to learning. It was developed in ScratchMaths project and has a clear focus on children's engagement with mathematics and programming. The '5Es' model consists of the following unordered phases that provide opportunities for students to: *1)* ***E****xplore* ideas by taking control of their own learning, debugging errors and looking for reasons behind different outcomes, *2)* ***E****xplain* and discuss their own ideas and being encouraged to use the programming language as a 'tool to think with', *3)* ***E****nvisage* by predicting outcomes before creating or running the code and then reflecting on and comparing the actual outcome with their prediction, *4)* ***E****xchange* and defend their own ideas and debug or build on others' perspectives, *5) bridg****E*** *by* making links between the programming environment and the mathematics through a reconstruction of ideas.

Based on the fundamental principle of constructionism, where students generate meanings as they construct or tinker with digital artefacts and discuss their mathematical ideas in relation to these artefacts and their behaviour (Kynigos, 2015; Papert, 1980) and considering that the TRU framework does not propose a specific way of teaching or learning with the use of digital technologies, in this research, the author tried to investigate the TRU framework with the use of a programming medium combined with the pedagogical framework of '5Es' in order to provide students with opportunities to generate meanings through a different way of interacting with the mathematical properties, compared to the siloed approach of the textbooks. Of course, there is not a strict one-to-one correspondence between the TRU dimensions and the '5Es' phases, as the TRU dimensions overlap and the '5Es' phases are combined at many points in the learning process. On the one hand, the dimensions of the TRU framework provide valuable aspects for teaching mathematics, as they are designed to be easily understood and open to the teacher or researcher. However, they wouldn't show a deep understanding of students' mathematical experiences or details of students' ideas during their interaction with the programming tool. On the other hand, the '5Es' phases provide this deeper insight into the mathematical experience through opportunities for students to engage with and construct meanings with and through the tool, but they remain focused on these elements and don't refer to the overall value of the learning process or the role of the teacher. Therefore, the researcher discusses the use of a combination of these two theoretical frameworks as a better perspective for the creation of a strong learning environment based on programming for the development of children's mathematical thinking and competence to use mathematics in different contexts and, thus, she addresses the following research question: How do students use mathematical properties and, in particular the concept of the variable, while they are tinkering with animated models in the programming environment of MaLT2?

DESIGN OF THE RESEARCH

The activity in the programming tool MaLT2

MaLT2 is an online tool for symbolic expression in mathematical activities through programming for the creation of 2D and 3D graphical models, recognized by the Greek Ministry of Education. It integrates the UCBLogo language with the affordance of dynamic manipulation of procedures with variables (Kynigos & Grizioti, 2018) and provides three interconnected representations: (i) symbolic, (ii) figural, (iii) dynamic manipulation of generalised variable values through sliders (Fig.2c). Its affordances allow the construction of animated models, where students have to apply mathematics in such a way that the models maintain their properties as the values of their variables change. The designer of the activity decided to take advantage of the mathematical properties that can be embedded in the construction of the Greek capital letters N, Z, Σ, M, A in a programming environment because, firstly, they are not typical mathematical shapes and therefore do not immediately reveal their mathematical nature to the students, and secondly, the construction or debugging of the letter models could be quite flexible, providing many entry points for the students to engage in the activity and opportunities to develop their mathematical thinking. Thus, the design of the activity was mainly based on the didactic design of "half-baked" microworlds (Kynigos, 2007), where buggy models 'hide' their mathematical properties behind their engineering way of construction and aim to engage students in tinkering and debugging them. The phases of the activity are shown in Table 1, each coloured differently and linked to the '5Es' framework during their design stage. However, the author only refers to those in bold, as it was during these activities that the episodes presented took place.

Activity	'5Es' phases
(A) Prediction & debugging of half-baked generalised model N	Explore – Envisage – Explain
(B) Find the code of a given animated model of N (in a square)	Explore - Explain
Free construction of an animated model of letter Z	Explore – Explain - bridgE
Half-baked words ΝΗΣΙ & ΑΜΜΟΣ (mean island & sand)	Envisage – Exchange – bridgE
Free construction of a dynamic poster with all letters	5Es

Table 1. A brief description of the activities and their design according to the '5Es' framework

Methodology of the Research and Data Collection

The research was conducted with four 9-grade students of secondary school who worked together in two groups in an after-school class. They participated voluntarily in the intervention, which lasted 8 teaching hours over 3 weeks. The students had already been introduced to the basic components of MaLT2. Each group had a laptop, a worksheet for each activity and a notebook. The researcher played the role of the facilitator during the intervention, encouraging students to express and justify their ideas, while data was collected from computer screen and voice recording, students' notes and MaLT2 codes from the letter models constructed by the groups. This study was a pilot study of a larger design-based research (Cobb et al., 2003). The data were analysed through the lens of the TRU dimensions and the '5Es' phases based on episodes (Noss & Hoyles, 1996) regarding the generation of students' meanings or the debugging of students' misconceptions about the use of the variable during their interaction with the models of the letters in the programming environment and their classmates or the researcher.

RESULTS

The researcher identified instances of students' discussion and actions concerning the use of the variable in the animated letter models in MaLT2. Table 2 presents students' actions related to the five dimensions of the TRU framework identified through the '5Es' phases. The following episodes took place during phase 1 of the activity and therefore (A) refers to the 1st activity presented and (B) to the 2nd one (see Table 1, in bold).

Dim. 1 The Mathematics Students made connections to previous concepts and procedures and developed a strong mathematical thinking, when:	• the role of the variable as a generalised number emerged after debugging each other's ideas *(Exchange)*, through the challenge of the activity to give the code, but not the execution values (A) • they thought and decided to use the concept of sine in their model *(Envisage - body sintonicity)* by thinking with the tool (B) • they realized the link *(bridgE)* between mathematics (i.e., the concept of the sine) and the programming environment (i.e., the specific procedure needed (B) to its use in the code)
Dim. 2 Cognitive Demand Students met challenges and engaged in a 'productive struggle' through:	• Their predictions *(Envisage)* and discussions *(Explain & Exchange)* that led them to take control of their own actions and learning by ***Exploring*** the model of a triangle with the sliders without being asked (A), by working on the misconception that x always plays the role of the unknown (B) and, gradually, by building on the perspective that x is used as a generalized number in the models (A & B) • The iterative tests with the digital tool during the *Exploration with the sliders, the 3D camera and the scene's grid* that led the students to realise that the given model of N fits into a square, to make sense of the concept of sine or the Pythagorean theorem in the construction of this animated model and, above all, to put these ideas in use in the code, which requires an engineering approach to mathematical properties (B)
Dim. 3 Access to Mathematics Active and meaningful engagement of all students through:	• The prediction *(Envisage)* of both the model (A) and the code (B) • The many entry points of the activities *(Explore)*, such as the non-unique answer (A - e.g., letter N or triangle) or the variety of mathematical properties that could be embedded in the letter models (B - e.g., Pythagorean theorem, trigometric numbers, ratio)
Dim. 4 Agency, Ownership, Identity Students developed their willingness to engage in the activity and the control of their creations by:	• *Explaining* their thinking and building on each other's ideas about the role of the variable: unknown vs. generalized number (A & B) • *Exploring with the sliders*, making conjectures about their models and the given variables (A & B) and gradually replacing stable numbers by variables (side dependence) & vice versa (A) Students developed a positive mathematical identity as doers by generating and sharing ideas about the use of properties in the generalised models and by correcting their misconceptions about the use of the variable (A & B) throughout all the '5Es' phases
Dim. 5 Formative Assessment The responsiveness of the environment to students thinking appeared through:	• The digital tool, which promoted the comparison between the students' predictions and the real outcomes *(Envisage)* and led to the active exploration of the limits of their current perception though the *Exploration* of their artefacts (A & B) • The researcher's question, which tried to address the students' emerging misunderstanding about the use of the same variable in all sides (B) of the triangle, without giving the answer, but by trying to engage children in a "productive struggle"

Table 2. An overall view of the presented activities' results through TRU and "5Es" Frameworks

The following episode took place during Activity A (Fig. 1a), where students were given a "mystery" code and asked to predict what it could construct.

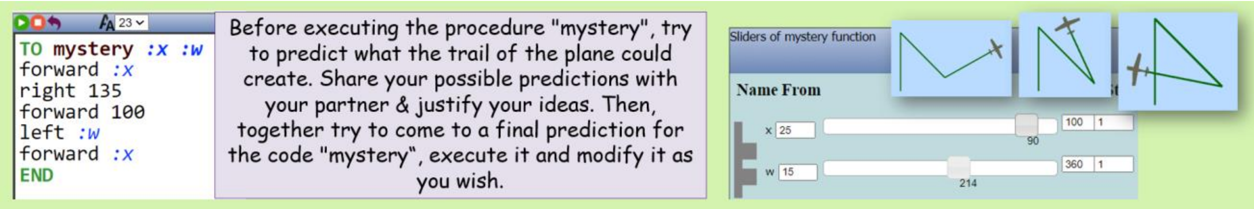

Figure 1. a. The given code for students' prediction, b. Screenshots of students' dynamic manipulation of the model through the sliders after the code execution

Student A: So, I think that it gives the letter N.

Student B: Yes, I agree, but wait. It goes forward x. The right command is fine. Then forward 100. It doesn't go forward that much.

Student A: What do you mean? It doesn't say how many steps forward the plane goes. It goes less.

Student B: How do you know it's going less than 100, since it doesn't say so?

Student A: Well, we only predict. But, yes… We don't know if it is more or less than 100.

Student B: Exactly. Forward x means the plane goes forward as many steps as we want. And then left w. Wait! Could it also be a triangle? … If the left command goes over 180 degrees, then this side (shows in notebook) turns a lot to the left and closes, so a triangle is constructed.

The crucial point in this episode is the children's discussion about the role of the variable in their predictions. At first, both students implicitly assumed that x was a certain number, so they seemed to use the letter x as an unknown. In fact, Student A stated that the letter x was a number less than 100, while Student B assumed a number greater than 100. It is interesting to note that even though both students had a certain number or range of numbers in mind for the variable x, they were at the same time aware of the uncertainty of this number, as they questioned each other. Through the expression of their thoughts, students made connections to previous knowledge about the role of the variable as a generalised number *(Dim.1 The Content)*, which emerged after debugging each other's ideas *(Exchange & Dim.4: Agency & Identity)* and both built on the perspective that x is not used as an unknown *(Dim.2: Cognitive Demand)* in the model ("as many steps as we want"). The key feature of this activity that triggered the students' discussion was the concealment of the execution values of N at the end of the code, a fact that seemed to encourage the students' willingness to engage in the prediction of the model and to be involved in a discussion about the role of the variable x. If the designer had given the execution values along with the half-baked model (e.g., x=50 w=120), the students probably wouldn't have made these different assumptions. Thus, the *Exchange* phase from the '5Es' model was prominent here, even though the researcher hadn't included it during the design of the activity. Furthermore, the opportunity to predict the model before executing the procedure seemed to challenge students to look deeply into the code, intuitively explore the possible outcomes and discuss the command that has a variable in the left turn that could actually construct a triangle (Fig. 1b). After their predictions, the students executed the code, compared their predictions *(Dim.5: Form. Assessment from the tool)* with the real outcome and explored the triangle model for some time using the sliders *(Explore)* before deciding to finally modify it to construct a generalised model of the letter N, using the concept of ratio.

In contrast to Activity A, in Activity B, students were given an animated model of N and asked to create a procedure (code) that could construct this model. The key feature of Activity B was that this letter model fitted into a square. After *exploring* the model N using the sliders, the 3D camera and the scenes' grid, the students identified the square and one of them thought about using the concept of sine in the code (Fig. 2a).

Teacher:	And how did you think of using the sine in the code?
Student D:	When I saw that two triangles were created in the square, in this triangle, I went and stood on the top, here (shows on the screen), and from there the plane was looking across. So, when I saw the opposite side, I also thought about the hypotenuse and that's how the sine came into my mind.

Even though the students had the figural representation and were looking for the code that constructed this model of N, the *Envisage* phase had also worked the other way round. Student D mentally put himself in the place of the airplane (Fig. 2a) and imagined the path and the movements that this avatar had made, by means of their body (‘*body syntonicity’* - Papert, 1980), using two different verbs to describe his thought (“I stood – The plane was looking”). This possibility, offered by the programming tool, allowed the student to generate the idea of the sine *(Dim.4: Agency)* by making connections to previous knowledge and procedures *(Dim. 1: The Mathematics)*. However, the students’ actions and the discussion that followed, showed that they had difficulty using it in the code because of a misunderstanding concerning the concept of the variable.

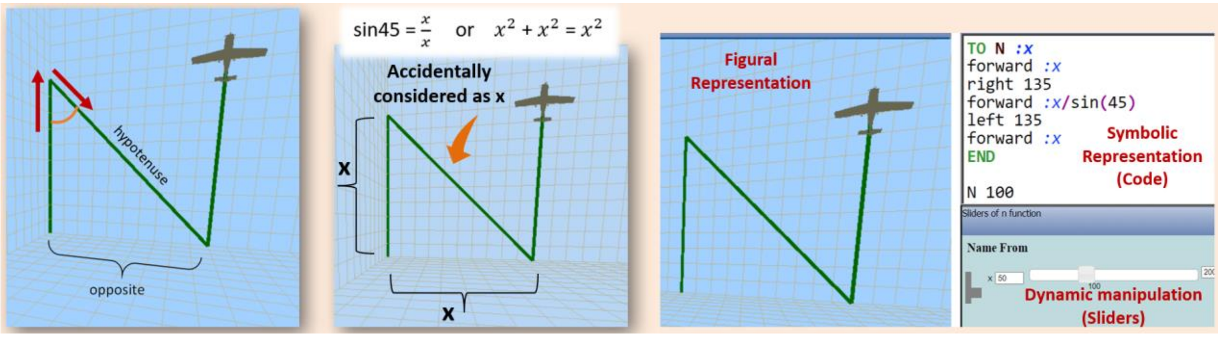

Figure 2. a. Student’s thought about the sine through body syntonicity, b. Students’ misunderstanding and tests, c. Final code & model construction using correctly the concept of sine in the command ‘forward’

During the children’s repeated testing on the notebook and the tool, and before they reached the final code shown in Figure 2c, the following episode occurred.

Student D:	Something is wrong… It doesn’t make sense, and it doesn’t work either with the sine that I thought of firstly or with the Pythagorean Theorem that we tried afterwards.
Teacher:	Which side is x?
Student C:	The hypotenuse, that is what we are looking for.
Student D:	(after 1 minute…) Wait! What did we set as x?
Student C:	The hypotenuse.
Student D:	Oooh that’s it! We put x on the hypotenuse, but the other 2 sides are x!
Student C:	I don’t understand!
Student D:	In our model the two vertical sides are x, right? Because of the square outside the N. You see? Forward x! The hypotenuse we are looking for is not x.
Student C:	Yes, that’s right. It’s more than x, because it’s a lateral side.
Student D:	Right. In our notebook, we also put it as x, because at school we use x for whatever we are looking for.
Student C:	That was like an explosion in my head! Let’s try it again!

In this episode, the responsiveness of both the digital tool through the instant feedback and the teacher's reflective question "Which side is x?" *(Dim. 5: Formative Assessment)* who had immediately understood the students' misconception about the variable, acted as a facilitator in the learning process of the use of the variable x. Even though the students initially seemed to be able to justify the choice of the sine in their code (Fig. 2a), they appeared rather confused about how to use it in the code command, where they set all sides as x and made also the same mistake when they tried the Pythagorean theorem, instead of the sine (Fig. 2b). Students engaged in a 'productive struggle' *(Dim.2: Cognitive Demand)* and managed to make sense of the different role of the variable through repeated tests from the paper to the tool, indicating a rather serious problem of the school curriculum, which follows a siloed approach to many concepts and has led students to the misconception that an unknown concept is always represented by x ("…at school we use x for whatever we are looking for."). Thus, based on this episode, the researcher could claim that the students reconstructed their ideas between the way that mathematical concepts and properties are used in the programming environment of MaLT2 and the way they express these ideas in 'official' mathematics *(bridgE)*, as the use of the variable is different in the command 'forward :x' (generalized number) and in the hypotenuse of the triangle (unknown), as the students did. In addition, the fact that the students remained positive in testing their ideas and checking their model over and over again, allows the author to assume that they built up a positive identity as people who can do mathematics. *(Dim.4: Ownership & Identity).*

DISCUSSION

The results presented showed that the students' mathematical identity was fostered as they became designers, used mathematical properties and discussed the concept of the variable and its use in a programming environment where mathematics took on more of an engineering substance. The groups used the concepts of ratio, sine and the Pythagorean theorem to construct or debug their letter models, in an engineering way rather than the usual school way, which gave them the opportunity to express their thoughts about the variable in the animated models. The episodes showed that the children approached the variable mainly as an unknown number that they will determine at some point. On the one hand, they tried to think of the variable as a generalized number, but it seemed that in their minds they actually had x as a specific number during Activity A. However, through the prediction of the model, the students debugged and at the same time built on each other's arguments, reshaping their knowledge on the variable as a generalized number. On the other hand, during Activity B, the students directly expressed the procedural process they have been following in school where x was always what they were looking for, without even thinking about it. However, the digital tool gave them the opportunity to develop a critical attitude towards the use of mathematics in a different context. The affordance of dynamic manipulation of the models by changing the variable values and the instant feedback offered by MaLT2 were crucial to the activity, as they encouraged students' repetitive cycles of exploring, assuming and testing. This encouraged them to make connections between the different uses of the variable, and ultimately to feel competent in using mathematics as a tool for thinking, constructing and reasoning. More than that, the use of the TRU framework, enriched with digital technologies and combined with the '5Es', enlightened the analysis of the students' meanings and ideas about the use of the variable as a generalized number and the study of their mathematical experience as a whole in a programming environment. Besides, the appearance of all the TRU dimensions illustrated a powerful learning environment based on the designed activities using the '5Es' phases and the MaLT2 digital tool. Thus, the author could suggest that these theoretical perspectives may provide an insightful way of designing activities for students, aiming not only to teach 'new' mathematical concepts, but also to develop students' competence and agency to use the mathematics that they already know through a different approach and in different contexts. Clearly, there are limitations and the results of this research cannot be generalised. However, it seems worthwhile to further explore the combination of these theoretical approaches as a better perspective for creating strong learning environments based on programming for the development of students' mathematical thinking and competence, also exploiting 3D dynamic models.

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