EXPLORING THE ROLE OF 3D PRINTING TECHNOLOGY IN SUPPORTING UNDERGRADUATE STUDENTS' TOPOLOGICAL CONCEPTUAL KNOWLEDGE

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In this paper we explore the educational potential of using a 3D printer in a university mathematics setting. We investigate how 3D printing technology might help undergraduate mathematics students improve their conceptual knowledge and theoretical reasoning. Our approach intends to engage students in both the design of topological objects and their 3D printing manufacture as well as the study of their properties through the manipulation of phisical examples. Extending the "example space" and moving from one semiotic representation to another are fundamental learning opportunities. The former is crucial in reifying a mathematical concept by including even a material example, and the latter is critical in activating cognitive processes to acquire conceptual understanding. Preliminary findings highlight that the 3D printing technology effectively integrated students' learning experiences and creativity.

Keywords: 3D-printing, concept understanding, example-space, topology, university mathematics

INTRODUCTION

In this age of digital transformation, 3D printing technology is being used successfully in a variety of industries, including manufacturing, medical, engineering, aerospace, and science. 3D printing is based on digital fabrication (DF), which is described as "the process of translating a digital design developed on a computer into a physical object" (Berry et al., 2010, p. 168). Using 3D printing in STEAM and STEM education improves students' creativity, cooperation, problem-solving skills, and higher-order thinking skills, as well as impacting their interests, engagements, beliefs, and careers, according to reviews (Ng, 2017; Cheng et al., 2021). In recent years, there has been a growing interest in employing 3D printing technology to improve mathematics education. The focus is on the impact that 3D printers can have on teaching and learning mathematics, as well as their potential for both knowledge construction and creativity development. According to a recent study (Stigberg, 2022), 3D printing is the most often used approach for creating manipulatives that reify mathematical concepts in geometry, algebra, functions, and fractions. The use of 3D printing helps students visualize concepts and proofs (e.g., geometry, calculus, volume) and enables them to develop mathematical, abstract, and spatial thinking (Dilling & Witzke, 2020). It not only scaffolds students' mathematical understandings, it is also a powerful tool to stimulate students' creativity and spatial and design thinking (Ng, 2017; Ng & Ferrara, 2020; Medina Herrera et al., 2019). Ng and Ferrara (2020) report how primary students used 3D printing pens to create their own prisms and pyramids to learn the geometric properties and cross-sections. A physical 3D-printed mathematical object is simply one example of a mathematical concept. Research argue that engaging students in the production of examples is a necessary step to understanding definitions and overcoming difficulties in facing advanced mathematics problems (Watson & Mason, 2005; Dahlberg & Housman, 1997; Meehan, 2007; Moore, 1994). Difficulties appear to depend on the delicate transition between seemingly conflicting approaches, instrumental-relational or operational-structural (Skemp, 1976; Sfard, 1991), to grasp a concept and apply it to learn others and solve problems. According to Sfard (1991) an abstract notion can be conceived in two fundamentally different ways: *structurally,* as objects, and *operationally,* as processes, and these two approaches, although incompatible, are complementary. The transition from computational operations to abstract objects is a long and inherently difficult process, and consists of three steps: *interiorization, condensation,* and *reification.* In the first stage a learner gets acquainted with a concept and performs operations or processes on mathematical objects, then he/she has an increasing capability to alternate between different representations (Duval, 2017) of a concept, and finally can conceive of the mathematical concept as a complete, "fully-fledged" object. Tall and Vinner (1981) relate the understanding of a concept to the distinct notions of "concept image" and "concept definition". The former describes the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall and Vinner, 1981, p. 152) and it is built up by individual through different kinds of experiences and associations with the concept's idea. The latter is "a form of words used to specify the concept" (p. 152) and refers to a formal definition. Moore (1994) integrates the *concept image/ concept definition* construct with the notion of *concept usage*, "which refers to the ways one operates with the concept in doing proofs" (Moore, 1994, p. 252). According to Moore there are different ways to use a definition: generating and using examples, applying definitions within proofs, using definitions to structure proof. Concept definition, concept image and concept usage define a *concept-understanding* scheme (Moore, 1994). As we pointed out, encouraging students to analyse conventional examples, or produce examples their own, helps to deepen concept understanding and facilitates the discovery of proof. Watson and Mason (2005, p.51) say that "examples can be perceived or experienced as members of structured spaces" and introduced the term "example space " to describe such space. They consider the extension and exploration of example spaces as an essential element in learning mathematics:

> Learning mathematics consists of exploring, rearranging, and extending example spaces and the relationships between and within them. Through developing familiarity with those spaces, learners can gain fluency and facility in associated techniques and discourse. (p. 6).

Watson and Mason (2005) distinguish between "personal example spaces" (PES) and "conventional example spaces". The former refers to a repertoire of available examples of concepts, and methods of example construction, for their own personal use, coming from personal experiences. The latter refers to set of examples conventionally used by teachers and displayed in textbooks. The constructs of learner generated examples (LGEs) have been developed and described by Watson and Mason as a powerful pedagogical tool, that is, a tool which helps learners to recognise, appreciate and more deeply understand general principles. Examples help to overcome the difficulties (Moore, 1994; Dahlberg & Housman, 1997; Meehan, 2007). Dahlberg and Housman (1997) suggest it might be beneficial to introduce students to new concepts by requiring them generate their own examples, starting from a concept definition and going to enrich its concept image. Gallagher and Infante (2019) define *structural examples*, characterized by a more abstract nature than *concrete examples* (specific, generic, algorithmically generated), and examine through this new construct students reasoning about definitions in introductory topology. A structural example is an example that possesses only the essential properties of a definition but lacks any additional properties that might be present in a concrete example. In particular, 3D-printed example of a topological object is a concrete example stimulating the abstract production of a structural example. The 3D-printing is the last step of a process which involves the passage, through a mathematical software, from one semiotic representation of the object to produce to another. The entire process is worthy of study by referring Duval's theory of semiotic representation (Duval, 2006). Indeed, students pass from an analytical representation to the corresponding graphic representation and, from the latter to its physical one. We conduct a preliminary study on the use of 3D printing technology with the aim to reduce the students' gap between the cognitive construction of the concept associated with a mathematical object and the real construction of a physical object representing it. Students' construction of examples by themselves contributes to achieve the familiarity useful to an aware learning, and to develop problem-solving affective, cognitive, metacognitive competences. In particular, to be engaged in these 3D-printer constructions help students to visualize concepts and proofs and to discover new ones. The practice also has immense power to increase motivation and satisfaction, with a highly probable increase in ability to solve real problems.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

The project foresees students' engagement in problem-solving activities stimulating cognitive and metacognitive processes: thinking, designing, producing, and then analyzing, argumenting, defining, conjecturing, proving, reflecting. In a first phase, in a digital environment students create and explore by themselves the objects to be studied by fingers in a second phase. The problem-solving activity aims to foster the acquisition of concepts and to promote the generation of new mathematical knowledge, through emphasizing conceptual rather than procedural learning in specific digital learning contexts. We follow Schoenfeld's (1985) problem-solving view, according to which a mathematical task is a problem to explore, aiming to understand a concept and to find a strategy not already known to solve it. The reification (Sfard, 1991) of a mathematical concept can be related to the construct of 'concept understanding' including those of 'concept image', 'concept definition' and 'concept usage' (Tall & Vinner, 1981; Moore, 1994). In our context the concept understanding scheme also refers to another aspect: the digital manipulation and production of an object. We could speak of "concept digital manipulation". Concept digital manipulation could be seen as a kind of digital mediator between the 'concept image' and 'concept usage' as it refers to the digital experiences such as the digital production of a graphical representation and the digital fabrication of a material object in a specific learning context (digital and empirical). We are interested in objects produced by the digital manipulation of concepts and in their usage. These objects are examples enriching the example space, graphical to see and physical too handle. 3D-generated example are associated to both personal example spaces and to conventional example spaces (Watson and Mason, 2005). Duval's theory help us to investigate the gap between the mathematical objects that we intend our students to build cognitively and those that, in reality, the student constructs. Transformations of semiotic representations are placed at the center of the mathematical activity. According to Duval, transformations of semiotic representations are at the heart of the mathematical way of working and to teach it is crucial to develop students' cognitive skills to match different representations (Duval, 2017). To view the same object in different representations and to create correspondence between objects or between representations is crucial for developing the cognitive processes for knowledge. In the conversion process we start from the analytical-algebraic representation and pass through the software to the geometric-graphic one and through the 3D printer to the geometric-physical one. Each representation of the concept is an occasion for reflection in which the treatment transformation can give even a partial answer to the mathematical problem. Another aspect we focus on is creativity. Creativity is the essence of mathematics and cannot be separated from technology The lens through which we will seek creativity will refer to research that enhances the idea that doing mathematics is not only procedural and that convinces the student that each problem does not have exactly one correct answer or only one correct path to arrive at the solution. The analysis will therefore pass through fluency, flexiblity (Leikin, 2009) and originality. Occasions to bring out students' creativity manifest at various times on two levels. The former relates to designing objects, and its nature is rather operational, while the latter realizes first through digitally manipulating objects, and then exploring by hands and conjecturing about the corresponding 3D-printed representations. Both add a more structural dimension to the learning experience.

Research questions

This polyedric research theme stimulates curiosity in various directions opening the door to at least as many questions: digital competencies, problem-solving competencies, creativity, discourse, inclusion, construction of conceptual knowledge, development of theoretical thinking. We investigate how 3D-printing technology affects the construction of mathematical knowledge (concepts, definitions, theorems) and stimulates of cognitive processes. We ask the research questions: *What is the impact of the of 3D-printing technology on the construction of mathematical knowledge. Specifically, what is its role on stimulating cognitive processes activating in the construction of mathematical knowledge? Does it help students to reify a topological concept?*

METHODOLOGY

The context

The research is part of a wider project concerning the role of examples in the development of advanced mathematics competences through problem solving activities at university level: going from the exploration to understand a concept or a theorem to defining, conjecturing, proving. It refers to generating examples in topology, to be planned for fifty students attending a third-year course of the bachelor's degree Course in Mathematics. The experience will realize within an introductory course of Algebraic Topology, in which we give definitions of algebraic objects that are topological invariants, such as the fundamental group, and we open the discover of the classification of topological surfaces through both fundamental group and the Euler-Poincaré characteristic. In this context the idea to engage students in activities aimed at extending example spaces through 3D- printing would add another dimension to the learning experience, and increase motivation and curiosity. Students, as designers, print and build up an artifact, and the final product is a physical object they explore and classify by hands. A physical object produced through computer design is an example. We conducted a preliminary pilot study to test the potential of the idea. Three graduating university students in Mathematics were involved in a problem solving preparatory activity dealing with the fabrication and classification of topological objects. The problem developed in a first phase as a combination of thinking, design, and production and in the second phase as a manipulation of the 3D-printed objects to discover their properties. In line with the design, the students, divided in groups (only a group in our preliminary study) were assigned to print some spaces to explore, going from reconstructing their analytic representations to exploring it, through sliders, and, then, to printing. In the conversion process we started from the analytical representation and pass through the software to the visual-geometric-graphic one and through the 3D printer to the visual-physical one. The final product is a physical object. Each representation of the concept was an occasion for reflection in which the treatment transformation already gave an answer, even partial, to the mathematical problem. The design foresees that a student is both a designer of the mathematical objects and a solver of the mathematical problem dealing with them. Students design, explore, print, manipulate, classify, generalize, in a digital environment made by the software Mathematica and a 3D-printer.

A sample of task

The project aims to involve the student in problem solving activities focused on the student's design of the digital object and on the exploration by fingers of a mathematical properties and relations. As an example, in the first task question students explore metric properties obtaining different objects but topologically the same object. How far can they go? Let's take an one-sheet hyperboloid to shrink until to get a cone. The spaces are no longer homemorphic because one has a trivial fundamental group and the other does not. Students collect printed objects and open the discussion on the properties of a mathematical object in order to classify it topologically, even by calculating the Euler Poincaré characteristic. A particular attention is paid to the classification of surfaces. In the second task-question through a set of printed objects (Figure 3) they pass from graphic exploration by sight and parameters to tactile exploration by fingers. The actual touching of the physical objects can lead to "new gestural forms of thinking". The task foresees undergraduate students' engagement in the design, in a Mathematica digital environment, and printing of 3D topological objects. The project plans to involve students both in the design andvprinting of the topological object and in the study of its properties by fingers. There are two problem-solving levels related to the actions to design and to solve, both improving theoretical thinking and creativity through exploration and transformation of different semiotic representatios. Students were assigned the following kind of task:

Task

1. Design and print a set of mathematical objects (hyperbolas, cone, Moebious strip,…)
2. Study, compare and classify that objects through the topological properties discovered manipulating them during the fabrication process

Data collection

We have at our disposal two types of data: written notes by the students, notes about the discussion. We focus our analyses of the role of 3D-printed examples on students' processess in understanding mathematics and in developing creativity both in a personal and a collective situation, looking at the students' written and oral discourses and at the teacher's transcripts of the discussions.

Data analyses

Because of space restrictions, we confine our analysis to a significant part of the discussion transcribed by the teacher during the last phase of the learning path experienced by the students, wich focuses on the results that emerged in the problem-solving process after the 3D printing of the objects: the discourse about discovering the topological properties of an object and conjecturing about them.

PRELIMINARY FINDINGS

In the problem solving activity we ask students to digitally design and produce, through a 3D-printer, material objects, as results of semiotic trasformations (Table 1) .

Topological space	Representation 1 Parametric Mathematica	Representation 2 Visual-Graphic	Representttaion 3 Physical-3DP

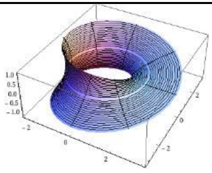
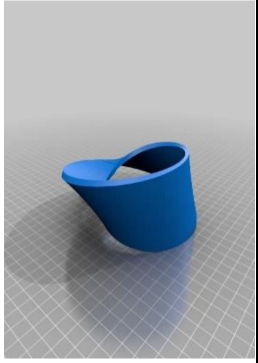
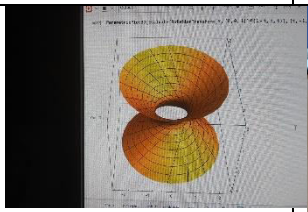
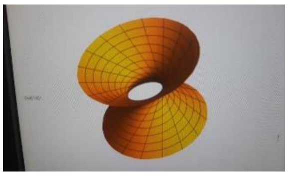
The Möbius strip	x[u_, v_] := (1 + (v/2) Cos[u/2]) Cos[u] y[u_, v_] := (1 + (v/2) Cos[u/2]) Sin[u] z[u_, v_] := (v/2) Sin[u/2] plot = ParametricPlot3D[{x[u, v], y[u, v], z[u, v]}, {u, 0, 2 Pi}, {v, -1, 1}, Boxed -> False, Axes -> False] Double rotation	 	
The one-sheet hyperboloid *Two models*	x[u_, v_] := aCosh[u]Cos[v] y[u_, v_] := bCosh[u] Sin[v] z[u_, v_] := c Sinh[u] plot = ParametricPlot3D[{x[u, v], y[u, v], z[u, v]}, {u, 0, 2 Pi}, {v, 0, 2 Pi}, }, Boxed -> False, Axes -> False]	 	

Table 1. From analytic to physical 3D-printed representation of a topological space

Then we ask to create correspondences between different objects (topological classifications, as an example) or between other representations (Table 2), in accordance with Duvall's theory theoretical framework of registers of semiotic representation. Here (Table 1, Table2) a sample of answers of the previously described task requiring to describe, to visualize and to expolore by Mathematica, to 3D-print, to explore by finding. A taste of this last phase is given in the following transcript.

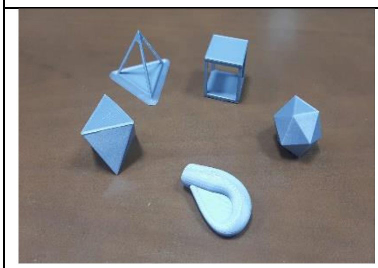
A set of 3D-printed Topological spaces	Classification 1 Metric type	Classification 2 Topological type	Classification 3 Homotopic type
	Five equivalence classes Why?	Three homeomorphism classes Why?	Three homeomorphism classes What if we took the base off the tetrahedron?

Table 2. Classifying a set of a topological spaces by handling them

In the discussion, students handle 3D-printed models of two different one-sheet *hyperboloids* (see Table 1) and a cone. They explain they feel the objects and deduce properties and correspondences. On the basis of the experience, the topological properties are described pre-formally and further justifications are given and written. For example, the symmetry of the hyperboloids, or the compute of fundamental group. The following discourse describes students' working with the 3D-printed models.

Student 1: [moving the hyperboloids models around a finger]They both are rotational symmetric to my finger, the axis!!

Student 2 discovers properties of objects by handling it:

Student 2: And my hand a plane with this slope [putting her hand in the middle of both] intersect one in a circle and the other in a small circle!! They are not isometric. But have them other simmetries?

Student 1: How did you do that? We want to know if they are homeomorphic.

The reflection of Student 3 concerns the sensual experience of the characteristics of hyperboloid in relation to the possible actions on their graphs, in accordance with Tall (2013).

Student 3: Yes they are!! You can imagin to restrict one to obtain the other. But.. Mhh. Look at this! [Places the finger around the object in the middle] When you put your finger on it like this, you obtain a cone. Imagine exploring by Mathematica sliders… Is it again homeomorphic?

Student 2: Oh no!! The homeomorphism we have described identifying the one sheet hyperbolas is no longer good! This is one point! [Places the finger around the the middle] When you restrict here, you obtain a point. Is there another one? I don' know.

After a teacher' hint to discover if some topological property distinguish the surfaces

Student 3: All are compact, connected. Compute the Eulero Poincaré characteristic,…, it's the same,..mhh

Student 1: No, they aren't!! The cone is contractible. You can put continuously it in the vertex.While the cylinder have the same (up to isomorphisms) fundamental group of the circle.

The sensory handling of the models represented an important component of the students' reflections. Students to classify a set of spaces (Table 2) argued based on feeling the shape and the holes of the 3D-printed objects The explorations, after all, revealed that students developed a competent discourse about topological properties and relactions, and that the technology integrated their learning experience.

CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

Recently, digital technology occupies a place of honor to support the teaching and learning of mathematics, as an integral tool to stimulate rationality and mathematical thinking. In mathematics education, 3D printing is an innovative way to visualize and handle mathematics concepts that enables students to develop theoretical and design thinking, as well as digital skills. Moreover, this allows students to develop fluency that opens up more opportunities as a digital designer, as a problem solver, as a researcher. Our research project foresees to involve students first both in the design of topological objects and in their 3D-printing, and then in study of their properties and relations through fingers. The educational potential is undisputed, both in terms of digital skills and in terms of mathematical skills. Students develop cognitive processes going to the acquisition of mathematical knowledge. Extending the space of examples with a material example, allows students to reify a mathematical concept, and therefore to develop theoretical thinking and creativity. 3D- printing is a tool to access the mathematical objects and to establish correspondences between them and between different forms of representations. The trasformations of semiotic representations are crucial to develop students' cognitive skills. Moreover, this practice fosters inclusiveness. Perception of the real world depends on the senses. And a blind student ore a visually impaired graduate student through touching is put in a position to be able to identify the geometric properties of an object like everyone else. A visually impaired graduate math student interviewed on the subject says that 'the use of 3D-printing would certainly have been useful, especially in the representation of homeomorphisms. […] it is much more easily clarified with the support of a visual example, tactile in this case'. The research opens up new perspectives for investigating inclusion and authonomy of visually impaired and blind students.

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