Sine, you think, it can dance? An Aesthetically Driven Mathematical Activity for Meaning Making on Trigonometric Functions

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This paper discusses the role of mathematical aesthetic experiences in the shaping of two 11th grade students' meaning making on trigonometric functions, while they engage in coding dancing figural models as music visuals for a specific song. MaLT2, the main digital resource used, is a programmable 3D Turtle Geometry software affording dynamic manipulation of variable values. GeoGebra was also used as a tool for plotting parametric functions. Students freely explored and manipulated functions of sine and cosine for animating their digital artifacts, starting from their simplest forms and building up to Fourier series approximations. They connected these functions to figures' length or angle temporal change, based on their periodic and symmetric dynamic features. Findings, though limited in scale, provide a glance into how an aesthetically driven mathematical activity in such digital environments supports students' meaning making and mathematical inquiry.

Keywords: Educational technology, programming, dance, trigonometry, aesthetics.

INTRODUCTION

Researchers have highlighted the importance of (re)considering the aesthetic aspects of school mathematics, based on the affordances of digital technologies (Papert, 1978; Sinclair, 2018; Bu & Hohenwarter, 2015). They argued that their expressive and sensory power has allowed students to experience mathematical aesthetics in a novel and accessible way, by integrating the factors of subjectivity and personal sensibility within its traditional autonomous, elitist perception (De Freitas & Sinclair, 2014). Nonetheless, aesthetic considerations related to mathematics learning remain marginalised and understudied in this research community (Sinclair, 2018). Following a humanistic perspective on mathematical aesthetics, as conceptualised by Sinclair (2004, 2018), we designed a learning activity that privileges aesthetics by ingraining mathematics in personal artistic creation. In this study, we traced and analysed students' aesthetic learning experiences (Papert, 1978; 1980) from their engagement with expressive digital media, where trigonometric functions were used and explored for the evaluation and creation of animated shapes, matched and tuned to a specific song.

MATHEMATICAL AESTHETIC LEARNING EXPERIENCES

Traditional approaches connect the mathematical aesthetic with objective values of mathematical beauty, mainly based on its rationality and purity of logical deduction within a platonic view. In this perspective, mathematical aesthetic experiences, that result from the ability to appreciate these aesthetic values, are accessible only to a small minority of people. This epistemological discussion has mainly focused on the fundamental role of aesthetics in mathematicians' practices during their creative processes of developing new mathematical content. Three main aspects emerged from this discussion; the role of aesthetics in 1) guiding the mathematician to discovery, 2) motivating the selection of certain problems and 3) helping a mathematician evaluate a certain result (Sinclair, 2004).

A growing number of researchers perceived the aesthetic as a mode of cognition and highlighted the importance of cultivating it in the mathematics classroom (Papert, 1978; Sinclair, 2018; Jasien & Horn, 2022). However, the elitist traditional perspective eliminates opportunities for most students from experiencing it. With traditional learning practices dealing with mathematics as a body of isolated knowledge detached from human senses, governed by deductive logic, where truths are already true, aesthetic experiences are completely neglected in schools (Sinclair, 2018; Jasien & Horn, 2022). Papert (1978) was strongly opposed to this objective view of mathematical aesthetics, that deprives students of experiencing it, by pointing out, both theoretically and practically, ways of designing aesthetically rich learning environments that allowed drawing, exploring and making of mathematics. He linked the notion of aesthetics in mathematics to aesthetics in the arts, through adding the subjective component of the 'sensible' (Sinclair, 2004). The embodied sensory engagement in mathematical inquiry, i.e. what is perceived with one's eyes, ears, skin, emotions and senses in general, introduces students to extra-logical facets in mathematical thinking. This type of engagement, that we define here as mathematical aesthetic learning experiences, connects mathematical meaning making to aesthetic practices and affective processes. According to Sinclair (2004), such experiences shape mathematical sense making in three ways, as adapted from considerations on the role of aesthetics in mathematicians' inquiry; through 1) a generative, 2) a motivational and 3) an evaluative role. The generative role involves the guiding process of gaining insight connected to both problem posing and problem solving. It is physically driven by feelings of wonder and curiosity that give rise to ideas on the formation of a particular problem or on the way to proceed with its solution. The motivational role refers to the development of personal interests that attract learners to engage in mathematics in particular ways. Having the freedom to select mathematical concepts, problems and strategies based on inner motivational mechanisms can lead students to develop a personal taste and agenda on mathematical inquiry. It is connected to emotions of interest and desire. Finally, the evaluative role concerns the learners' engagement in the process of deciding whether a specific result of mathematical inquiry is good or beautiful enough, following a socially shared or a personal set of criteria. It is connected to emotions of surprise, amusement, anger, confusion and disappointment.

DESIGN RESEARCH

The task and the digital resources

Two digital resources were used for the design of the task for different, complementary purposes; MaLT2 (http://etl.ppp.uoa.gr/malt2/) and GeoGebra (https://www.geogebra.org/calculator). On the one hand, MaLT2 was the main expressive medium for the creation of the animated artefacts. This software integrates a UCB-inspired Logo procedural language with Turtle Geometry in 3D and dynamic manipulation of parameter values through sliders (Grizioti & Kynigos, 2021; Kynigos & Karavakou, 2022). Users can construct figural models through programming for the movement of an avatar (turtle = hummingbird in 3D) that leaves a coloured trace behind. They can also animate them by defining a parametric procedure (e.g. '*TO shape :t*') whose parameter (*:t*) is included as input in a logo command (e.g. '*left :t*' or '*right 2*:t*' or '*forward 30*sin(:t)*') or in a sub-procedure (e.g. *dancer1 :t*). By dragging a slider, the values of the corresponding parameter change and the figural transformations of the avatar's trace are shown in the 3D scene (Fig. 1). By constantly pressing the keyboard's right arrow for moving its slider, a parameter can conventionally represent time, embedding the concept of motion in time. Based on its expressive, graphical and dynamic affordances, we considered MaLT2 as an appropriate medium for the creation of periodically animated figural models, matched both chronically and qualitatively to a specific musical piece. On the other hand, GeoGebra had a supportive role throughout the task. Two GeoGebra files were designed, used as graphing calculators for plotting i) trigonometric functions of the form $a_1 sin(b_1 t)+c_1$ and $a_2 cos(b_2 t)+c_2$ and ii) approximations of Fourier series of the form $a_1 sin(t)+b_1 cos(t)+a_2 sin(2t)+b_2 cos(2t)+a_3 sin(3t)+ b_3 cos(3t)+a_4 sin(4t)+b_4 cos(4t)$, where parameter values a_i, b_i could be manipulated through sliders.

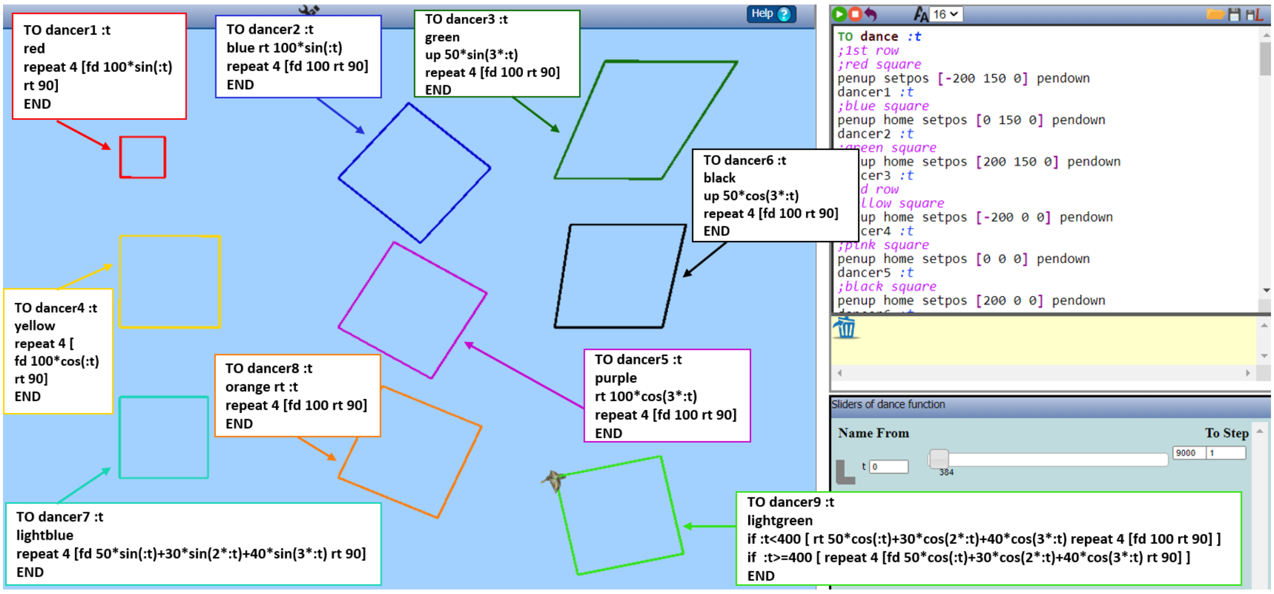

Figure 1: Screenshot from the constructed artefacts in MaLT2 for Phase 1

For the 1st phase, that had an introductory-preparatory status, we designed a ‘dance competition’ simulation in MaLT2. It consisted of nine squares moving simultaneously in a different periodic way through arrow-key-dragging of the slider for the parameter *t* of the ‘*dance*’ procedure (Fig. 1). As the values of *t* were steadily increasing, each square was moving periodically either by rotating or by fluctuating its sides’ length, with their period and way of moving varying depending on the trigonometric functions included in either a turn or a moving-forward command. For example, the red square (Fig.1, dancer1) was moving in a simple fluctuating way based on the command ‘*forward 100*sin(:t)*’ with a period of 360 (in t-values); the purple one (Fig.1, dancer5) was making a rotating motion based on the command ‘*right 100*cos(3*:t)*’ with the period of 120 and the light-blue one (Fig.1, dancer7) was fluctuating in a more complicated periodic way, based on a Fourier approximation of ‘*forward 50*sin(:t)+30*sin(2*:t)+40*sin(3*:t)*’, with a period of 360. A specific song with stable repetitive rhythm was chosen to accompany the ‘dancing’. Finally, a task sheet was designed, where all nine procedure codes where given nameless, in random order. The task included students working together on rating the nine different procedure codes with grades from 1 to 10, based on the corresponding square’s motion they believed was created by each one of them, as ‘dancing move’ matching (or not) to the song. For the 2nd and main phase, students were free to create dancing figural animations in MaLT2 for another given song with various rhythmic units.

Data collection and analysis

This study is part of a design research project, following its final adaptation phase before its main pilot classroom implementation. Two students of 11th grade, Petros and Maria, who had some previous experience with MaLT2 and GeoGebra, participated in a six-hour, divided into two days, out-of-school setting. The collected data included screen and voice recordings from one shared laptop, written task sheets and gestures noted down by the attending researcher during students’ activity. Data was analysed using coding while tracing for the generative, the motivational and the evaluative role of aesthetic experiences in mathematical meaning making, as defined in the previous section. At the next section, some concrete examples of situations where such aesthetic roles are evident in students' activity and discourse are described.

RESULTS

Phase 1: Evaluating the aesthetic code

During the first phase, students worked together in order to mutually agree on a grade for the evaluation of the code of each square's 'dancing move'. Even though this task had an introductory purpose, there were some interesting cases where aesthetic experiences played an evaluative and a motivational role to their mathematical inquiry. For example, while arguing on the evaluation of the first two rows of squares (dancer1-dancer6), students were led to comparison, argumentation, conjecture formation and testing in GeoGebra. During their inquiry, meanings on the period and input-output values of trigonometric functions came up:

Petros: I like sine! You get it? It goes right and the to the left because it gets negative values! (…) But it's unfair to put a higher grade here (dancer1). The green (dancer3) is better matching its tempo.

Maria: Yes, but the red one (dancer1) is making a nicer move. Even though it is too slow. I really like that when the music tone gets higher, the square is to the right and highest point and when the tone gets lower, the square is to the left.

Petros: Ok, but still. The green is so perfectly synchronized to the song. It's like circling together. We need to give a higher grade to the green one. Maybe 5 or 6 and 7?

Maria: The black square is not as synchronized as the green one. It seems though like the black one is a bit slower. (…) These two squares move in 3D and correspond to the commands *up 50·sin(3t)* and *up 50·cos(3t)*. I think they have the same period.

Students' aesthetics played an evaluative role for the judgement of each square's movement. They chose to use the issue of synchronicity as a main criterion for grading the codes. They started a process of comparing the periods of each animation in order to determine which codes are better matching to the song's rhythm. The motivational role of aesthetics led them to a series of negotiation and mathematical investigation serving their aim to be fair. In order to test his conjecture about the period of the first two rows of squares (dancer1-6), Petros turned to the first GeoGebra file and set the parameters in order to plot and compare the functions *10sin(t)* and *5sin(3t)*, then *5sin(3t)* and *5cos(3t)* (Fig. 2.1.) and then *5sin(5t)* and *10cos(5t)*.

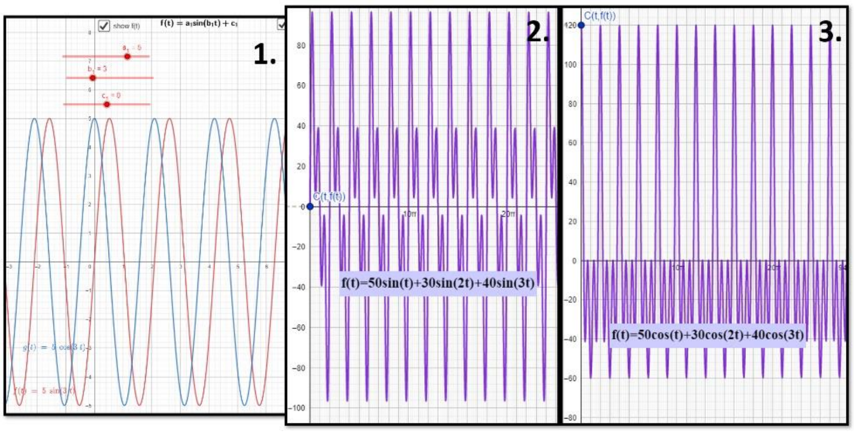

Figure 2: Screenshots from Petros's investigation in GeoGebra

Petros: Ok, it makes sense. When this $b=1$, the red and the blue squares are much slower, because, you see? The higher the number is, the thicker the graph gets. More ups and downs means quicker motion. The green and the black are 3 times faster. A whole period of the red corresponds to 3 periods of the black.

Maria: Between the green and the black, I think the green is better. Because it starts from being 2D, at position zero, and then leans forward and backwards, while the black starts by leaning. The starting point is important; it also matches the song better. And I think this applies to all codes including cosine, since it starts at the highest point, not at zero. So not at the neutral position as it happens with sine.

The evaluative role of her aesthetics led Maria to pay attention on squares' starting position. That motivated her to find a general evaluative rule for squares' starting position, that she based mathematically on the phase difference between sine and cosine and on their output values when their input value is zero. She made these last remarks while observing both the graph in GeoGebra and the figures in MaLT2 while setting the slider value to $t=0$. The evaluative and motivational aspects of aesthetic mathematical learning experiences were also evident during their evaluation of the third row of squares (dancer7, dancer9). The complexity of the squares constructed by dancer7 and dancer9 codes, caused by the approximation of Fourier trigonometric series as input of a logo command, led them to feelings of curiosity and wonder and further aesthetically driven discussion:

Maria: The light-blue (dancer7) is perfect. It should get a clear 10.

Petros: Are you sure? It is for sure a nicer move than the previous ones, but it's not perfectly synchronized with the song. It is not exactly sine and cosine, but it seems like having the same period as these ones (showing dancer1 and dancer4).

Maria: It is this one! Forward $50sin(t)+30sin(2t)+40sin(3t)$. Because the light-green one (dancer9) changes after $t=400$. So, its code is the one with 'if'. (…) It is not as symmetrical as the light-blue. I'd give 10 to the light-blue and 8 to the green. It seems to me completely out of rhythm, even though its move is nice.

Petros: Even though one is sum of sines and the other one sum of cosines with the same numbers, the first is symmetrical to the x-axis making a smoother move.

In order to support his decision, Petros got motivated to plot and compare the graphs of each trigonometric sum given in the task sheet in the second GeoGebra file (Fig. 2.2; 2.3.) and agreed on the rating of dancer7 to be 10 as the most symmetrical and smoothing one, with aesthetics having a motivational and evaluative role to his mathematical inquiry.

Phase 2: Making the aesthetic code

During the 2nd main phase, students carefully listened to the song they would make the dancing animation for. Their initial discussion included how each one pictured it, supported by reasons mainly based on emotions that the music brought them. During this brainstorming phase, the aesthetics had a generative and motivational role for setting up and pursuing goals related to features of trigonometric functions. For example, for the beginning of the song (first 24 seconds), one idea given by Maria defined their final creation:

Maria: What if we make a square fluctuating like the red one each time this part of the song is repeated? I think of a square going up and down and then at the second repetition another one, of different colour, appearing at another position, moving exactly the same. And then two others, following the same pattern. It will slowly raise the tension, at the same way the music does.

Petros: Yes! And when the beat drops, more shapes can appear all together! (…) The music is like constantly escalating, so more stuff can be added to render that.

Maria wrote the code of the *square* procedure and used it as sub-procedure in the *dance* procedure (Fig. 3.1.). She used the function *sin(t)* as an input to the *square* procedure, so that she would imitate the red square's (dancer1) dancing move from Phase 1. However, the result disappointed them, with their aesthetics having an evaluative impact on reconsidering the function in use.

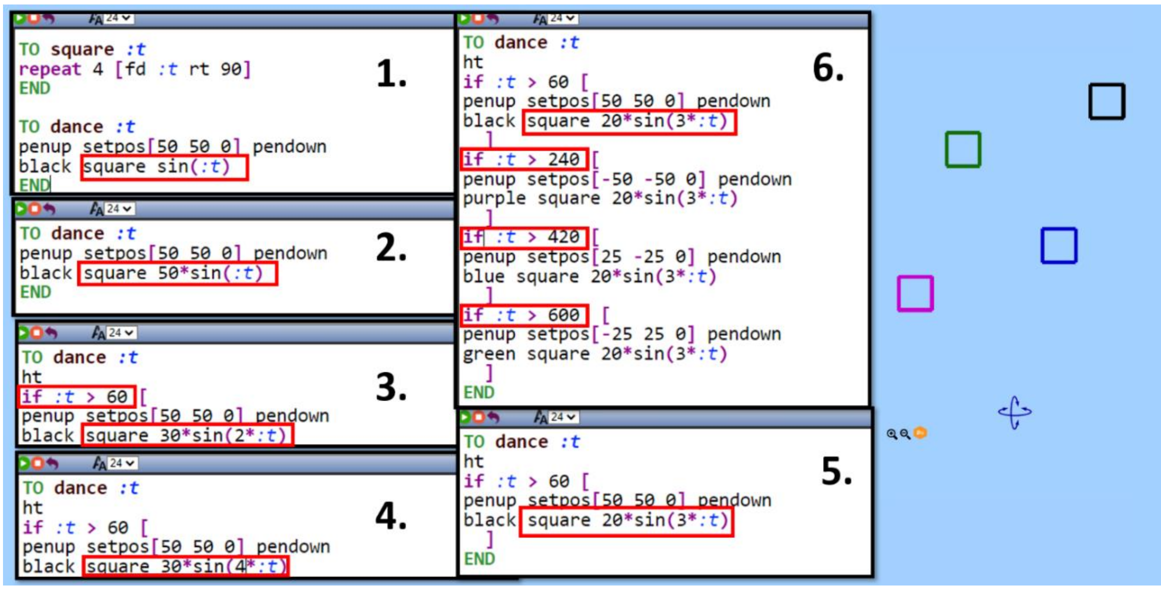

Figure 3: Screenshots from students' programming in MaLT2 in Phase 2 (Part 1)

Petros: Why doesn't it move? We put *sin(t)*, it was supposed to go like this, up and down.

Maria: Well, the red square (from Phase 1) was *100sin(t)*. We need this number in the front. But let's make it 50. The squares must be smaller to match the music, because at first it sounds very calm. (She changed the input function to *50sin(t)* as shown in Figure 3.2.)

Petros: No, but it is completely out of rhythm. It's too slow. We need to make it quicker.

Maria: We can change the number inside! To effect the density. But I want the square to be even smaller. Let's make it *30sin(2t)*? (She changed the input function to *30sin(2t)*, *30sin(4t)* and finally to *30sin(3t)* as shown in Figure 3.3.; 3.4; 3.5.)

Petros: Finally! No, the rhythm matches perfectly! Let's add the next square. Set its position at -50 -50 0, so to be symmetrical. (…) The second starts at 240 and the third at 420. It is a clear period, so at every 180 steps something is happening.

Maria: This means that the fourth square appears at *420+180=600*? Replay the song.

Petros: Perfect! That way we can add this tension that the music creates! (…) After *t=780* they can also be a bit quicker, or make a complex move, during this intense part.

As shown by the above parts of their discussion, the students went through a series of mathematical inquiry, guided by cycles of aesthetic experiences. It started with Maria shaping the idea of adding a small moving square every time a musical pattern is repeated, in order to slowly raise the tension, where her aesthetics played a generative role. Then it continued with both students motivational and evaluative aesthetic mechanisms used to fix the square's motion in order to be synchronized and match the song rhythm and tone, which led them to a circle of conjecture making and testing for the period and the dynamic representation of the sine function in MaLT2. Finally, the generative role of Petros's aesthetic experiences, boosted by the motivational and evaluative aspect of both students' ones, started a new circle of mathematical inquiry for reflecting the growing tension of the song, where they investigated, the use of sum of trigonometric functions in order to create more intense moves. Another example, where aesthetic experiences caused by the song played a key role for their mathematical meaning making process, was for the last part of the song, as partly shown below:

Petros: It gives me the feeling of stress. It has a growing tension, even more than before.

Maria: What if we'd make a hexagon growing and shrinking and then after only a few seconds another one doing the same move? As if it the second is chasing the first.

Petros: Yes, great idea! But how? (…) We can try $t+10$ inside each function, instead of t.

Students' aesthetic experiences originated from listening to the song had a generative role on the formation of the idea of 'chasing' as representing the feeling of stress. The motivational and evaluative aspects of their experiences led them to a lot of experimentation in MaLT2 and Geogebra, which ended up in the final part of the code (Fig. 4). Petros's following remark indicates the aesthetic influence over their choice of Fourier series approximation as input function:

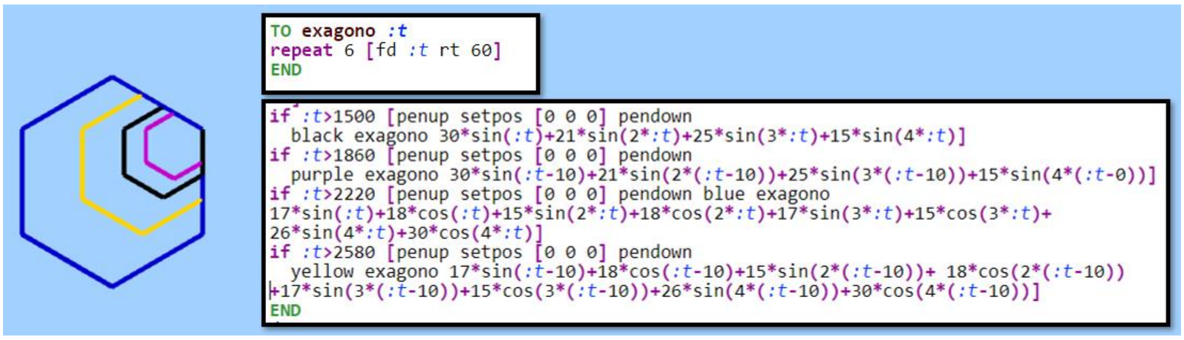

Figure 4: Screenshots from students' programming in MaLT2 in Phase 2 (Part 2)

Petros: At first we use a sum only of sines, starting in a smoother dancing move. Then as the song gets more chaotic, we tried a not very symmetrical one. So the 3rd and the 4th hexagons move with a more weird sum, including both cosine and sine.

Petros tested different combinations of sum parameters graphically in GeoGebra and then dynamically in MaLT2. He concluded that sum of sines of the form $a_1sin(t)+a_2sin(2t)+a_3sin(3t)+ a_4sin(4t)$ leads to a symmetrical move and intentionally avoid it by adding cosine factors of the form $b_icos(it)$ for creating the chaotic effect. This type of mathematical inquiry of setting out goals, posing problems and exploring their solutions using mathematical properties was guided by his motivational and evaluative aesthetic experiences. Throughout this process, he developed meanings on elements of Fourier trigonometric series approximation, such as symmetry, periodicity and its practical utilisation, as being able to represent different types of periodic variation.

CONCLUSIONS

The two students that participated in this study engaged in rich mathematical inquiry through cycles of problem posing, argumentation, conjecture formation and testing within both digital resources, appreciating the results and reconsidering the solution. These cycles were boosted by their aesthetic experiences, evident when they were listening to the song or observing the figural dancing animations in MaLT2. Throughout the six hour activity, we traced many cases in which their aesthetics played either a generative, a motivational or a evaluative role on their mathematical engagement. From these cases, that were viewed as mathematical learning experiences meanings, they developed mathematical meanings on trigonometric functions and Fourier trigonometric series approximation regarding periodicity, symmetry, input and output values, and intervals of monotonicity. They made sense on these concepts and relations and used them either to make a fair evaluation (Phase 1) or to create an aesthetically satisfying dancing animation (Phase 2). Both ways, mathematical meanings were generated naturally and gradually evolved through their expression and manipulation in the digital resources. Even though this paper involves a case study, it provides an example where students' aesthetics guided their mathematical learning experience, that was both aesthetically and mathematically rich in a proportional way. Our next research steps involve the adjustment of this study to the classroom level in order to gain a clearer general image on the role of aesthetics and expressive digital resources on mathematics learning.

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