MultiMap: A Computational Environment for Supporting Mathematical Investigations

Wallace Feurzeig, *feurzeig@bbn.com*
BBN Technologies

Paul Horwitz, *phorwitz@concord.org*
Concord Consortium

Abstract

MultiMap is a visual computational environment expressly designed to support the learning and teaching of mathematics. The MultiMap software transforms figures on the computer screen according to transformations or mapping rules (i.e., maps) specified by the user. The program enables students to design and experiment visually with maps specified by mathematical functions, algebra formulas, and geometric transformations and to investigate their properties and uses experimentally through an extensive set of constructionist activities.

A Brief Overview of MultiMap

MultiMap has a direct manipulation iconic interface with extensive facilities for creating maps and studying their properties under iteration. The user creates figures (such as points, lines, rectangles, circles, and polygons), and the program graphically displays the image of these figures as transformed by the map, possibly under iteration. MultiMap allows one to make more complex maps out of previously created maps in three distinct ways: by composition, by superposition, or by random selection of submaps. It includes a facility for coloring maps by iteration number, a crosshair tool for tracing a figure in the domain to see the corresponding points in the range, and a zoom tool for magnifying or contracting the scale of the windows, MultiMap also enables the generation and investigation of nonlinear maps that may have chaotic dynamics.

The program supports the creation of visual figures that are often ornate and beautiful such as self-similar mathematical objects of many kinds called fractals. The term “fractal” designates the convoluted curves and surfaces that exhibit self-similarity at arbitrary scales (Mandelbrot, 1983). Using MultiMap, with minimal guidance from an instructor, students have discovered such phenomena as limit cycles, quasi-periodicity, eigenvectors, bifurcation, fractals, and strange attractors (Horwitz and Eisenberg, 1992).

The MultiMap screen is divided into three windows as shown in the following figure. The user draws shapes such as points, lines, and polygons in the Domain window, using the iconic tools shown in the palette on the left. The computer draws the corresponding images of whatever shapes are drawn in the domain. The Map window specifies the transformation of points in the domain that “maps” them into the range. The user controls what the computer draws in the Range window by specifying a mapping rule, expressed in the form of a geometric transformation. The image specified by the map, drawn on the Range window, is computed for the entire plane. In the figure, the user has entered a rectangle in the Domain window and has then specified a map composed of two submaps, a scale and a rotation. Scale (0.8, 0.8) scales the rectangle to 0.8 of its original size in both x and y. Rotate (90 °) rotates the rectangle 90 degrees about the origin. In a

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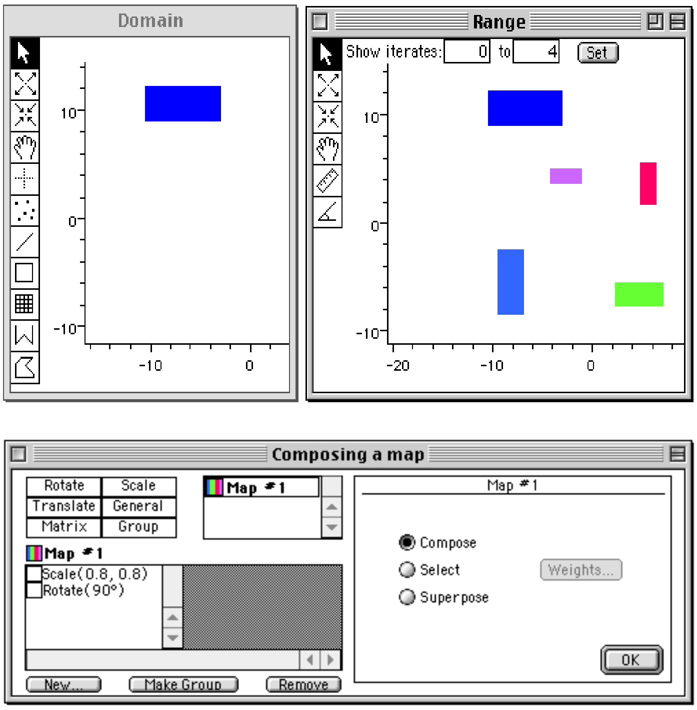
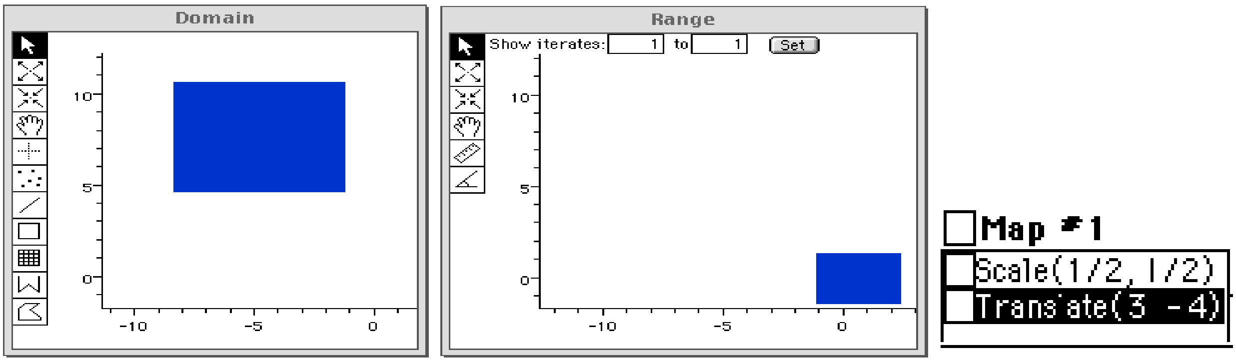
composition map such as this, the transformations are performed in order. Thus the rectangle is scaled and then rotated. This is an iterated map. The user has specified that the map is to be performed 4 times with a distinct color for successive iterations (light blue, green, red, and pink.) The Range window shows the result of the mapping.



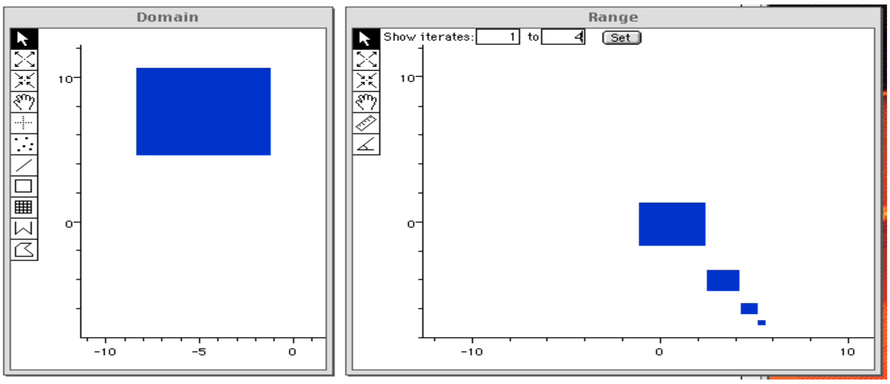
Iterated Scale and Rotation Map

Using MultiMap, students from local high schools created and investigated simple maps built on the familiar operations of rotation, scaling, and translation. Students were introduced to rotation, scale, and translation maps during their first sessions, and to their properties under composition and iteration. They then investigated the behavior under iteration of more-complex maps, including maps that produce beautiful fractals with self-similar features at all levels, random maps that generate regular orderly structures, and maps that, though deterministic, give rise to unpredictable and highly irregular behaviors.

The program has a direct manipulation iconic interface with extensive facilities for creating maps and studying their properties. The simplest geometric maps are created from primitive operations such as rotation, scaling, and translation. The user creates figures (such as points, lines, rectangle, and polygons) and the program graphically displays the image of these figures transformed by the map. The user also can specify a mapping function algebraically. For example, the function X' -> X/2 , Y' -> Y/2 will reduce figures to half their original size, and the function X' -> X + 3 , Y' -> Y - 4 will translate figures three units forward and four units down. MultiMap allows one to make more complex maps out of previously created maps. For example, the two functions described above can be composed (i.e., performed jointly) with the result shown below. As the figure shows, the rectangle that is input to the Domain window is scaled to half size and translated forward three units and down four units. The inset from the MultiMap control window shows the specified mapping operations.



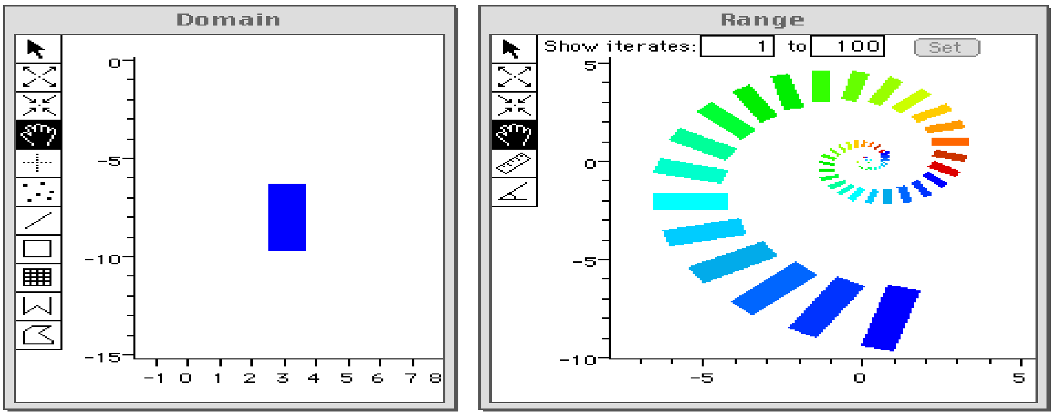
A user can repeat a mapping process an arbitrary number of times to generate a sequence of images. For example, when the mapping functions above are repeated four times in succession, the result is shown in the following figure.



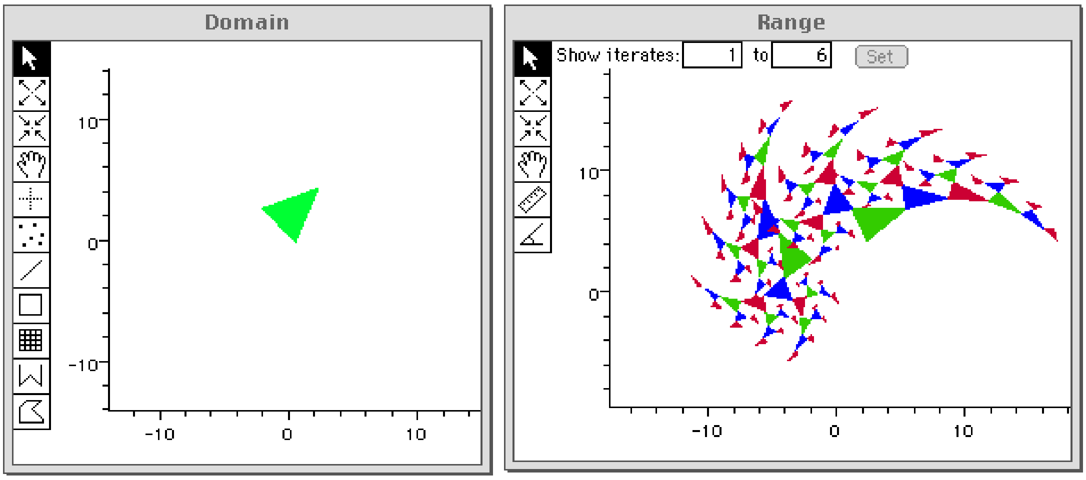
Iterated Scale and Translate Map

As the figure illustrates, MultiMap can display the limiting behavior of functions when they are iterated many times. Only one of three things can happen: successive iterates of the function may approach a single fixed point; they may converge to a limiting orbit of points; or they may behave more erratically, never quite returning to a value they have taken on before. Through investigating these situations, MultiMap can be used to provide students a clear and accessible introduction to sequences and limits, and a natural environment for investigating their behavior.

For example, students can create and investigate geometric sequences such as the following. (Color is used in these to show clearly the pattern of successive iterates, rather than for decorative effect, though it does heighten the aesthetic aspect of the mathematical structures.)



A Rectangular Spiral



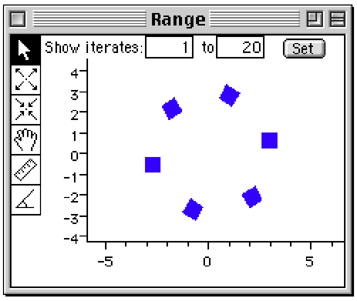
A Triangular Branching Pattern

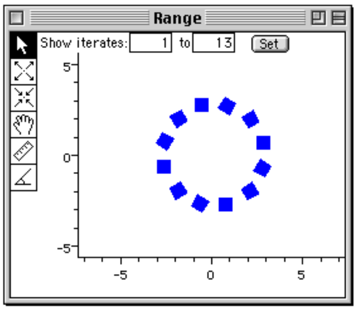
MultiMap connects the algebraic and geometric representations so that they are mutually supportive. The algebra helps students' understanding of the geometry, and vice versa. The software provides a variety of tools to aid exploration and investigation: a facility for coloring maps by iteration number (as illustrated above), a crosshair tool for tracing an input figure in the Domain window to generate the corresponding image in the Range window, a zoom tool for magnifying or contracting the scale of the windows, and a number of other tools to aid mathematical investigations. The program facilitates the creation of self-similar figures, and allows one to produce figures that are often very ornate and beautiful.

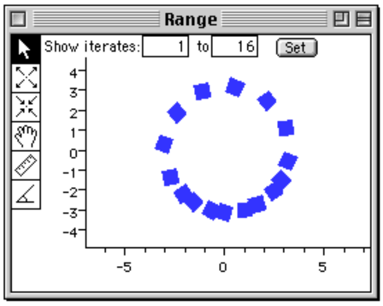
MultiMap enables students to engage in a rich variety of mathematical investigations. Its visual representations significantly aid in understanding function, iteration, algorithm, transformation, model and other key mathematical concepts. We have piloted the use of MultiMap with secondary students and teachers in algebra, geometry, and computer science classrooms and teacher institutes. The program has been used by over 50 math teachers who have demonstrated students' learning benefits and mathematical empowerment from working with MultiMap. Teachers find it easy to use and learn to write relatively complex programs. Through their work with MultiMap, students find that doing and learning mathematics can be *fun*.

A Student Session

The following session illustrates the use of MultiMap by two students, Kate and Fred, working together on an investigation of rotational symmetry (Horwitz and Feurzeig, 1994). They began by drawing a square and rotating it by 60 degrees, as shown on the left figure below. They noted that the 6 copies of the square lay around a circle centered at the origin, and that, though the map was iterated 20 times, after the first 6 iterations the others wrote over the ones already there. They were then asked what the result of a rotation by 30 degrees would be. Kate said that there would be 12 copies of the square instead of 6, no matter how many iterations. They confirmed this, as shown in the middle figure. The instructor then asked “What would happen if the rotation angle had been 31 degrees instead of 30?” Fred said “There will be more squares—each one will be one more degree away from the 30 degree place each time, so the squares will cover more of the circle.” MultiMap confirmed this, as shown in the figure on the right



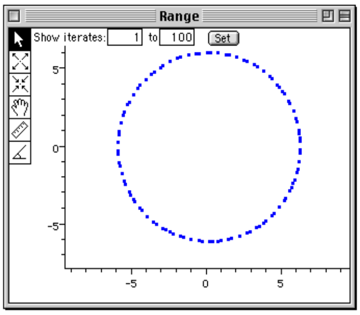


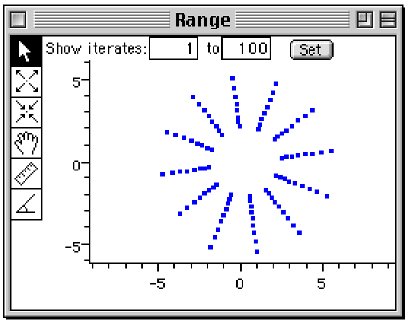


R(60) *R(30)* *R(31)*

Instructor: "*The picture would be less crowded if the square was replaced by a point.*" Fred made this change. The result, after 100 iterations, is shown below on the left.

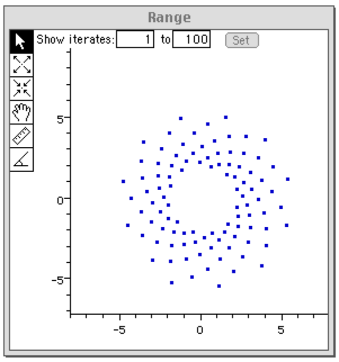
Since there was still some overlap, the instructor said "After each rotation let's scale x and y by .99. That will bring the rotated points in toward the center a little more at each iteration." Ann then built an R(30°)S(0.99, 0.99) composite map. The effect of the scaling is shown below on the right.

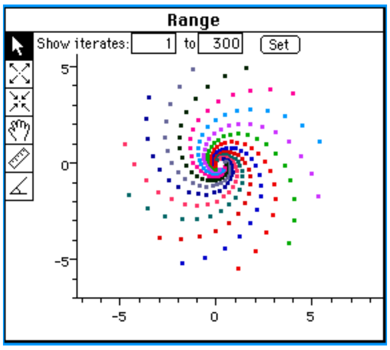




R(31) With Points *R(30)S(0.99, 0.99)*

Fred: "*Now the points come in like the spokes of a wheel with 12 straight arms*. The instructor then asked what would happen if the rotation were 31 degrees instead of 30.

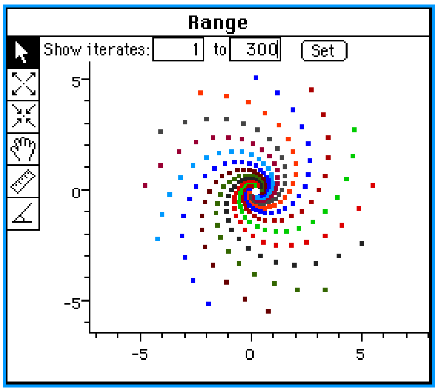


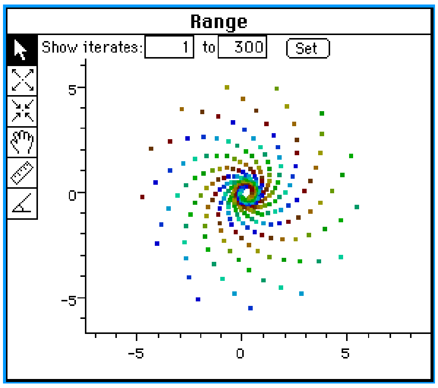


*R (31) * S (.99, .99)* *R (31) * S (.99, .99) 12-color ramp*

Fred replied "It would be almost the same but the points would not be on straight lines. He tried this. The result is shown on the left above. Kate said "The spokes have become spiral arms." When asked how many arms there were, she said "It looks like 12." The instructor said "Let's check that by making the points cycle through 12 colors repeatedly so that successive points have distinct colors." The result is shown above on the right. Kate: "Oh, how beautiful! And now each arm of the web has the same color." Fred: "Right, and we can clearly see that the web figure has 12-fold symmetry. Instructor: "What do you think will happen if the rotation is 29 degrees instead of 31 degrees?" Kate: "I think it will be another spiral, maybe it will curve the other way, counter-clockwise. But I think it will still have 12–fold symmetry. Here goes!"

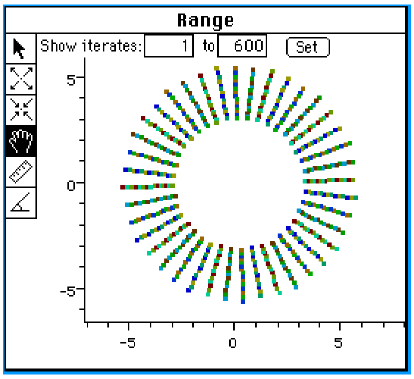
The result is shown below on the left. Instructor: "*Right! It goes counter-clockwise and it does have 12–fold symmetry. Very good! Now let's try a rotation of 27 degrees. What do you think will happen?*" Kate: "*I think it will be about the same, a 12–fold spiral web, maybe a little more curved." The result is shown below on the right.*

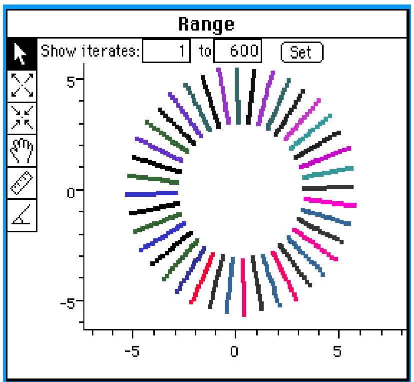




*R (29) * S (.99, .99) 12-color ramp* *R (27) * S (.99, .99) 12-color ramp*

Instructor: "*It might be that we don't have enough detail—let's get a more detailed picture by changing the scale from .99 to .999, and increasing the number of iterations from 300 to 600. See if that makes a difference*." The result, after 600 iterations, is shown below on the left. Kate: "*Wow, it looks very different now! There are many more than 12 arms, but they're all straight, and each arm still has many different colors*." Instructor: "*There's obviously much more than 12–fold symmetry here. Any idea what it is*?" Fred: "*120*." Instructor: "*Why do you say that*?" Fred: "*Because 360 and 27 have 9 as their greatest common divisor. So 360 divided by 9 is 40, and 27 divided by 9 is 3, and 40 times 3 is 120*." Instructor: "*What do you think, Kate*?" Kate: "*I don't know but I counted the arms and it looks like there are 40*." Instructor: "*Let's see if that's right. Reset the color map so that the colors recycle every 40 iterations instead of every 12 iterations*." The students changed the color ramp. The result, after 600 iterations, is shown below on the right.





*R (27) * S (.999, .999) 12-color ramp* *R (27) * S (.999, .999) 40-color ramp*

Kate: "*Now each arm is the same color. So there is 40–fold symmetry*." Fred: *Is 120 wrong?*

Instructor: "*No, 120 isn't wrong but it's not the only or the best answer. 240 and 360 would work and so would any other multiple of 120. But the real question is: what is the smallest one? The way to view the problem is this: what is the least number of times you have to go around a circle in 27-degree increments to come back to where you started? Or, to put it another way, what is the smallest integer N such that the 27 times N is an exact multiple of 360? The answer is 40 because 40 times 27 equals 1080, which is 3 times 360. No integer less than 40 will work." Fred: "I understand. Now I can do the problem for any angle*."

Investigating the Mathematics of Fractals and Chaos

We have begun to investigate the use of MultiMap on a rich variety of topics including the mathematics of chaos, fractals, and nonlinear systems. We seek to develop a coherent conceptual framework for introducing the key ideas at a level appropriate for high school presentation. To this end we are creating software tools designed to aid students in carrying out mathematical experiments and explorations. These tools will enable students to build and run models of dynamic systems with complex behaviors, to see their effects unfold, and to manipulate and study the generated graphic structures in multiple representations and at multiple levels of detail. We have started to design learning activities centered on the use of the tools and to develop organically the knowledge needed to use them effectively.

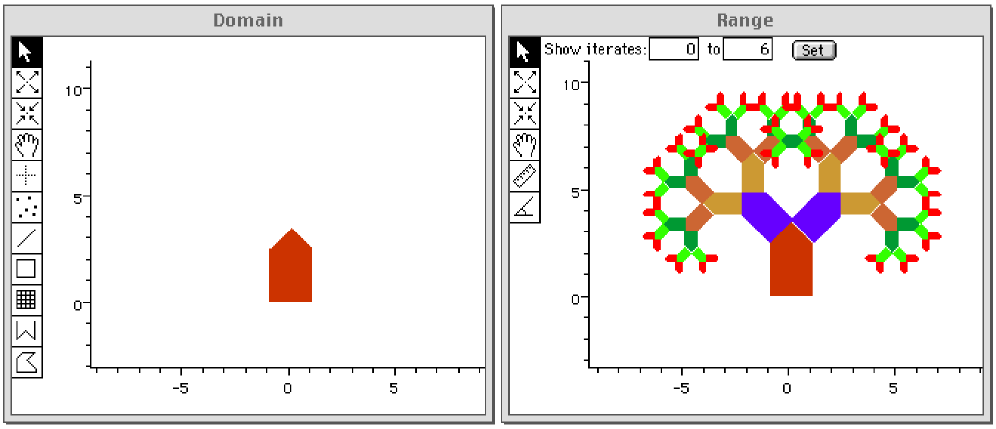
We believe that a nontrivial introduction to the ideas and methods of chaos can be developed and presented in a way that is both accessible and compelling to a significant fraction of pre-college students. This material is ideally suited to give students authentic experience of what doing mathematics and science is really like in areas that are meaningful and truly interesting to them. It provides rich opportunities for successful mathematical exploration, inquiry, and discovery. We plan to generate projects in relatively uncharted areas where it is possible for students to make new findings. In introducing students to the concepts and techniques of mathematical chaos we are placing them in a position to conduct investigations in a manner quite analogous to that employed by professional mathematicians.

Despite its modernity and complexity, an introductory presentation requires little mathematics beyond high school algebra. Moreover, the animated visual displays of chaotic processes greatly facilitate understanding of the deep connection between chaos and fractal geometry. The graphic pictures that are generated as natural outputs of investigations are often breathtakingly beautiful objects in their own right — the connection between mathematics and visual art has never been so apparent.

We introduce the subject of mathematical chaos to students by first familiarizing them with three fundamental concepts: iterated functions, maps, and fractals. Students then explore a wide variety of applications of chaos, e.g., to classical mathematical problems such as finding the roots of an equation; to the modeling of non-linear systems, such as the growth and decline of animal populations, the spread of infectious disease, the beating of the human heart, and the creation of fractal art and music. The use of MultiMap enables students to gain insights from visually rich mathematical explorations such as investigations of the self-similar cyclic behavior of the limiting orbits of rotations with non-uniform scaling (Horwitz and Feurzeig, 1994) and a better understanding of the deep issues underlying the solution of polynomial equations by generating maps that relate alternate representations of mathematical universes such as quadratic polynomials (Feurzeig, Katz, Lewis, and Steinbok, 2000).

The phenomenon of chaos is intimately linked to the behavior of functions, often very simple ones, when iterated many times. Only one of three things can happen: successive iterates of the function may approach a single fixed point; they may converge to a limiting orbit of points; or they may behave more erratically, never quite returning to a value they have taken on before. In the last case the iterated function sometimes displays an extremely sensitive dependence on initial conditions, so that neighboring starting points, when operated on repeatedly by the function, diverge very rapidly from one another, and all information about the starting point is lost. Behavior characterized by such an extreme sensitivity to initial conditions has been termed chaotic. The successive values taken on by the function closely resemble a random sequence, and indeed chaotic functions can be used as pseudorandom number generators. Because of their sensitive dependence on initial state, mappings of chaotic functions often display nearly self-similar structure on an infinitesimal scale, giving rise to curves and surfaces of fractional dimension, or fractals.

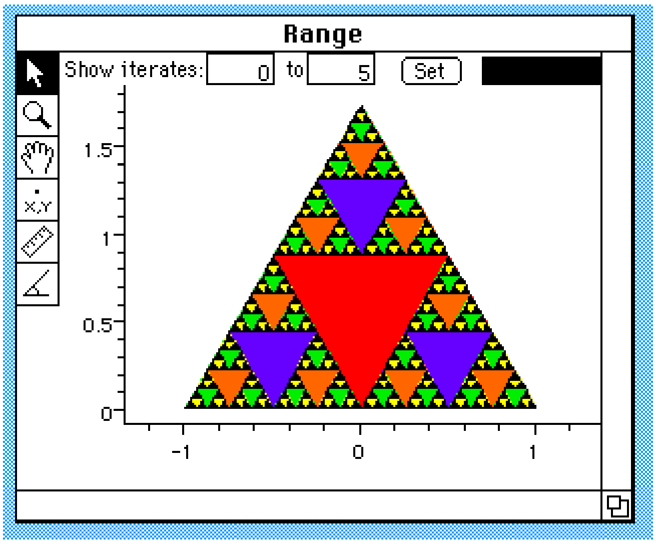
Fractals depict the convoluted curves and surfaces that exhibit approximate self-similarity at arbitrary scales (Barnsley, 1983). They can represent realistic images of natural objects such as flowers, clouds, and mountains. They can be amazingly complex and are often very beautiful. Fractal structures can be thought of as having non-integral dimensions. By virtue of its ability to generate recursive maps, MultiMap becomes a kind of "Fractal Construction Set" that enables students to create, modify and investigate fractals as objects of interest in their own right, even before they discover the deep connection of fractals with the phenomenon of chaos. For example, objects such as the fractal tree shown in the next figure, the result after several iterations of building scaled (doubled) rotated copies at each iteration, starting from the basic generating figure shown in the Domain window.



Generating a Fractal Tree

MultiMap supports recursive maps. It can map window A onto window B and then map window B back onto window A. This makes it a valuable tool for the study of iterated functions. For example, students can use MultiMap to construct pictures that contain "infinitely many" reduced copies of themselves. Such pictures can be constructed simply by creating a reduced scale mapping from one window to another, and then mapping the second window back onto the first, appropriately positioned. The iteration of these "condensation maps" often results in the creation of pictures that mimic such naturally occurring objects as ferns and clouds (Barnsey, 1986). In addition to being inherently interesting to students, these pictures illustrate the important idea of invariance under a scale transformation — an idea that underlies the concept of a fractal.

The following two figures illustrate the application of iterated maps for generation of fractal structures in MultiMap. The first one shows the "Sierpinski gasket", the result after three iterations of building successively compressed and three-fold multiplied copies of an embedded triangular pattern.



Sierpinski gasket

The generating figure, the initial iterate, is the large red triangle. The first iterate comprises the three blue triangles; the next two iterates are the nine orange triangles and the twenty-seven green triangles.

From High School Algebra to Chaos

Iterated maps are also useful in traditional mathematical activities, such as finding the roots of equations. One such application is to Newton's method, a well-known iterative procedure for locating the roots of equations in the complex plane. It can serve as an alternative to the quadratic equation formula routinely taught in high school algebra. It has the additional advantage that it can be generalized to finding the roots of cubic and higher-order polynomial equations, and that it can be motivated and justified to students via an appropriate graphical representation.

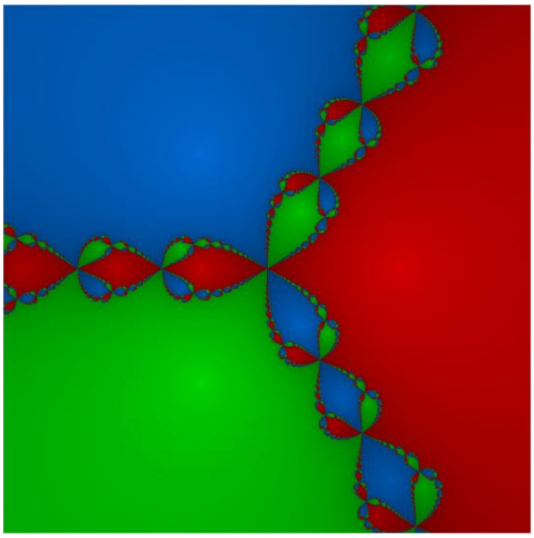
We introduced Newton's method in the context of quadratic equations, with which students were already familiar. We presented it initially merely as an alternative to the usual, somewhat mysterious formula. The method starts with an initial guess and then employs the repeated application of an algorithm that ultimately converges on one or the other of the two roots. We then posed the question: how does the choice of the initial guess determine the future behavior of the process? In particular, which of the two roots does the process ultimately converge on, and which initial guesses, if any, will result in its never finding a root? To answer this question, students began by using MultiMap to determine by trial and error the regions in the complex plane for which starting guesses converge to one or another of the roots of the equation.

For quadratic equations the solution is not surprising: connect the two roots by a straight-line segment and construct the perpendicular bisector of this segment. Then the "basin of attraction" of each root (that is, the set of all initial points for which the method converges to that root) is simply the open half plane on one or the other side of the perpendicular bisector. Points on the bisector itself do not converge to either root and in fact their behavior is chaotic, in the sense that the behavior under iteration of neighboring points diverges very rapidly, so that all information relating to the initial point is lost. In modern terminology, the perpendicular bisector comprises the so-called Julia set of the iterated rational function that characterizes Newton's method.

This new kind of exploration, in which one asks about the behavior of an iterated function at each point in the complex plane, requires a new kind of software tool — one capable of producing a variety of new kinds of mappings. The most obvious mapping simply assigns a different color to each pixel on the screen depending on the behavior of the iterated function at the corresponding point on the complex plane. Thus, a natural map of the situation described above is to color all points in the basin of attraction of one of the roots of the quadratic equation red, say, and of the other, green. This procedure divides the plane into two equal colored regions, separated by a straight line.

We then show students how to generalize Newton's method from quadratic to cubic equations, and give them the task of mapping out the basins of attraction of each of the three (complex) roots. Students expect the plane to be divided into three distinct regions, corresponding to the basins of attractions of the three cubic roots, just as the plane separated the basins of the two roots into two distinct regions for the quadratic equation.

However, the resulting map behaves very differently. It generates an extremely complicated and quite unexpected fractal picture as shown in the following figure. The reason for such remarkable behavior is simple. It can be rigorously shown that in the neighborhood of the Julia set (that set for which the function "cannot make up its mind" which of the three roots to converge to) there must be points belonging to each of the three basins of attraction at any level of iteration. In geometric terms: coloring the roots (say, red, green, and blue) at any point where any two regions (say red and green) come together, the other (blue) region must meet both of them, as well! There is no root-free boundary separating the regions. The structure is a fractal whose inner structure is repeated at finer and finer scales. MultiMap can demonstrate this strange phenomenon. Before representing the map, however, and after some consideration of this startling explanation of its behavior, the user may well have come to the conclusion that this situation is impossible. It is *not*, as the following figure depicting its behavior shows.



Cubic Roots Fractal

The observation of such astounding behavior motivates an introduction to the study and investigation of mathematical chaos. MultiMap provides users a powerful tool for experimental investigation of chaotic maps, those where the sequence of points generated by iterating the map exhibit "exquisite sensitivity" to initial conditions. (Horwitz and Eisenberg, 1992) describe and illustrate several such activities.

The properties of simple functions iterated many times are wonderful, unexpected and beautiful, but they may be expected to fall outside the set of inherently interesting topics for most high school students. To someone for whom the solving of equations — even beautiful ones — is not particularly motivating, the fact that this task can be accomplished through iterating a simple function is unlikely to be of lasting interest. It is important, therefore, to move on to activities in which the iteration of a function implies something more than merely finding the roots of an equation.

An obvious choice, and one that has rich mathematical and scientific applications, is to model a variety of processes that evolve in time. Each successive iterate of the function may be taken to represent a fixed time interval. If this interval is long enough to produce significant changes in the variables the resulting equation is a finite difference equation; if it is short on this scale, it approximated as a differential equation. Without the computer and an accessible tool like MultiMap, it would be unrealistic to attempt to introduce differential equations to the high school mathematics curriculum. However, once one has made a connection in students' minds between iterating a function and modeling a time-evolving process, the transition becomes natural and compelling, especially when introduced in the context of real-world situations such as prey-predator interactions, the spread of contagious diseases, and environmental recycling strategies. Many other areas of application are rich candidates for student projects with MultiMap.

The development of MultiMap in the NSF project "Advanced Mathematics from an Elementary Viewpoint" has been described by Feurzeig, Horwitz, and Boulanger (1989). Early versions were implemented on Macintosh desktop and laptop systems. We are currently implementing a new version for tablet systems such as the Apple IPad together with an additional body of project-based activities and supporting curricular materials.

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