Nicodemus explores Egyptian fractions: A case study

Ioannis Papadopoulos, ypapadop@otenet.gr
School of Primary Education, Aristotle University of Thessaloniki

Abstract

In this paper a fifth grader is involved in interplay with a computational environment trying to construct algorithms relevant to Egyptian fractions. More specifically, the student worked in the Balance environment which is a software aiming to help teaching and learning of rational numbers. In the context of this study the student was trying to find algorithms for expressing any unit fraction as sum of other unit fractions. This environment supported the student's experimentation and gradually he altered the initial interface of the Balance by adding or substituting components reaching thus a final version that could be regarded as an instance of an artefact that mirrored his own constructed knowledge.

Keywords

Egyptian fractions, Balance software, constructing algorithms, primary education

Introduction

Two aspects relevant to the issue of construction of knowledge are examined in the theory of constructionism. First, students learn by actively constructing new knowledge rather than by having the knowledge officially provided to them (i.e., constructivism). Second, what is taking place is learning-by-making, which means that effective learning takes place when the student constructs personally meaningful artifacts (Harel & Papert, 1991). The artifacts themselves constitute expressions of mathematical meaning and at the same time students continually express meanings by modulating them. In this spirit, the attempts of a fifth grader to construct algorithms—ways for writing any unit fraction as the sum of other unit fractions in the environment of *Balance* (a computer interactive program)— are examined. Thus, in a broad sense, the final product of the student's interaction with the program that would describe the asked algorithm could be considered as an instance of constructionism. A keyword in this process of reaching the algorithm will be 'experimentation'. Papadopoulos and Iatridou (2010) describe the systematic approaches of two 10^{th} graders who use experimentation to explore mathematical relationships, make and check conjectures and generalizations. They emphasized the importance of experimentation as an innate factor of successful problem solving. However, in this study a much younger student, a 5^{th} grader, is experimenting in a computational environment in order to discover algorithms concerning Egyptian fractions. More specifically, instead of presenting the algorithms to the student, the student himself interacts with the program, changes continually the given situation by adding or/and substituting components and ends with a situation that actually describes visually the algorithm.

The mathematical topic.....

It is known that Egyptians used fractions. More specifically, with the exceptions of 2/3 and 3/4, all of their fractions were unit fractions (i.e., fractions where the numerator is one and denominator any whole number). Thus, any fraction with a numerator larger than one had to be written as the sum of unit fractions. In order to carry out computations with unit fractions the Egyptians used to use tables. For example, they created a table of expansions of the numbers 2/n for all odd numbers n<100 showing the combination of unit fractions resulting from doubling unit fractions: 2/n=1/n+1/n. There is no unique way to write a fraction as a sum of unit fractions (Clawson, 1994). However, in this paper we are interested in algorithms for writing a unit fraction as a sum of unit fractions. According to Eggleton (1998) there is a unique answer to the question: 'In how many ways can 1/n be expressed as the sum of two positive unit fractions?'. If we consider d the number of positive integers that divide n^2 and ignore order in the representations then it can be proved that the answer is (d+1)/2 ways.

Numerous papers had examined the way fractions are conceived by students and had recorded relevant difficulties, misconceptions and errors. A full presentation of the research findings is beyond the scope of this paper. An important part of the existing literature describes the work done in a computational environment. Technology can provide an alternative to rote learning and automatic memorization since it can support students by allowing the construction of definitions and algorithms by students (Yerushalmy, 1997). So, we are asking for computer tools different from the drill and practice or tutorial software that are prevalent in many elementary schools. For example, Olive (2000) presents the TIMA that was developed in the context of a constructive teaching experiment focused on children's construction of fractions.

...the *Balance* program.....

The *Balance* is an interactive software that was designed in the context of *Enciclomedia*, a national project in Mexico. Its main purpose was to help in teaching and learning of rational numbers. It functioned as a space in which students and teachers could explore their ideas about rational numbers by working with activities involving equivalent fractions. The users can create balances with different numbers of weights and on different levels. On each weight, natural numbers, fractions and decimal numbers can be written. The program indicates, in real time, visually and with sounds, where the balance is in equilibrium or not, according to the values which are assigned to the scales. Working with teachers, Trigueros and Garcia (2005) found that the *Balance* can help teachers to reconsider their strategies and to understand the purpose of the activities included in the official textbooks relevant to equivalence of fractions. Working with students, Lozano and Trigueros (2007) found that the tool helped the students in their learning of the concepts related to fractions. More specifically, they found that the students gradually modified their actions from trial and error to finding systematic methods to solve the problems which included the use of operations with fractions and comparison of fractions using the concept of equivalence. Additionally, it seems that during this interaction between students and software mathematical learning occurs and this can be partly attributed: (a) to the fact that the program gives immediate and useful feedback inviting students to reflect on their own answers and to the fact that the students are provided with freedom to explore different situations, and (b) to experiment with different strategies (Sandoval, Lozano & Trigueros, 2006).

In this paper we try to broaden the usage of this software by asking students to construct an algorithm. Actually, the students are asked to proceed in an unorthodox way compared to the one they are accustomed. In the classroom usually a valid statement is presented and the students are asked to accept it and develop the relevant skill by working on a sufficient number of exercises. But, in this work, the wording of the statement (i.e., there are certain algorithms for writing a unit fraction as sum of unit fractions) is rather presented as a problem and the student's final step will be to invent these algorithms.

Description of the study

Nicodemus is a 5th grader in a primary school in Thessaloniki, Greece. He has been taught during the regular schooling basic facts about fractions: comparison, equivalence, operations. For the purpose of the study a web based flash version of *Balance* was used available at http://recursos.encicloabierta.org/enciclomedia/matematicas/enc_mat_balanza/ . Nicodemus was asked to find if possible at least two ways for writing a unit fraction as sum of unit fractions. Capturing software was used (CamStudio-Recorder) to record in a movie format anything happening on the computer screen. The student spent enough time to be familiar with the software before proceeding to the main activity that lasted one class-period. The student was asked to vocalize his thoughts while performing the task and the session was tape-recorded. Then the data were transcribed for the purpose of this paper. The movie and the transcribed protocol were examined in order to find out how the student's interaction with the tool can result to the creation of an instance of the tool that will support him to establish the asked algorithm.

The wording of the task was: "Use the *Balance* to find out at least two different ways for writing any unit fraction as sum of other unit fractions". Implicitly, the task highlights another important issue in mathematics teaching in relation to constructivism. While very often the support for constructivism comes from observations of situations where new knowledge has arisen from concrete situations, it is also necessary for constructivism to account for the more complex mathematics formed by the processes of abstraction and generalizations of earlier ideas (Booker, 1992). Thus, in the specific task it was expected that two algorithms could be found via the usage of the software: First, the pretty much obvious algorithm of writing each unit fractions as sum of its two halves: $\frac{1}{n}=\frac{1}{2n}+\frac{1}{2n}$. Second and more complicated was the splitting algorithm based on the equality $\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)}$. In terms of elementary mathematics this could be explained by the usage of equivalent fractions:

$$\frac{1}{n}=\frac{n+1}{n(n+1)}=\frac{n}{n(n+1)}+\frac{1}{n(n+1)}=\frac{1}{n+1}+\frac{1}{n(n+1)}$$

Results and Discussion

The examination of the student's attempts to construct the algorithms allowed us to split the whole process into distinct episodes.

Episode One: First Algorithm

Nicodemus found it very easy to set-up the first algorithm. It was not necessary for him to use the *Balance* as a vehicle to find the algorithm.

N1: I can use halves. For example if I have ½ then I can write it as the sum of its two halves 1/2=1/4+1/4.

He used the *Balance* only to verify his first algorithm (Figure 1).

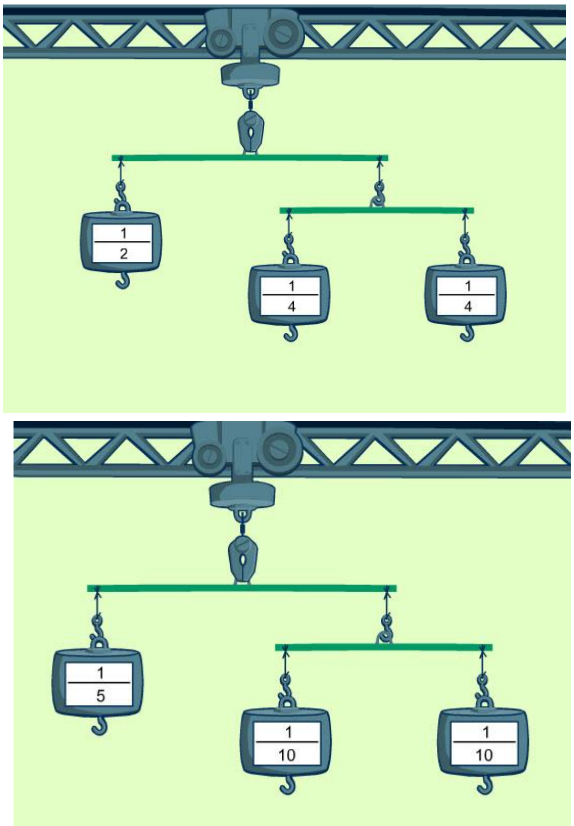

Figure 1. Using halves

He then used two more examples to show visually that his algorithm works: (a) 1/5=1/10 + 1/10 and (b) 1/9= 1/18 + 1/18.

At this point he made his first generalization:

N2: Each unit fraction can be written as the sum of its two halves which are also unit fractions.

Then, he wrote some additional examples in his notebook and again verified them in the *Balance* environment.

Episode Two: Second Algorithm – from unmethodical to systematic experimentation.

Nicodemus decided to work with the same initial unit fractions as in the first algorithm. He mentioned that he had to avoid using the two halves. So, he started with ½ on the left side of the balance and an arbitrary unit fraction on the right side. He had now to put an additional weight in the right side to achieve equilibrium. His choices were guided by the feedback he received from the software. Since his aim was to have in the end a horizontal bar in the upper level he started guessing and checking and then altering the denominators of the second fraction in order to correct the situation and to finally obtain a pair of unit fractions that would have sum equal with the unit fraction on the left side. However, since his experimentation was not following a concrete strategy he wasted his time by wandering around the unit fractions aimlessly. He very soon realized that it was not possible to find the algorithm this way. So, he decided to make a shift in his approach starting a new attempt starting with the fraction 1/5.

N3: I have to avoid using two halves.

N4. So I must start on the right side with a unit fraction smaller than the initial one on the left *side and at the same time different from its half.*

N5: I know how to make smaller or bigger a fraction but this is not enough to guess the correct pair.

N6: So, I can start on the right side with a fraction smaller than the initial and I will gradually start to change the second fraction of the pair in a constant rate to find the correct denominator.

N7: The first smaller fraction than the initial (i.e., 1/5) is 1/6.

Obviously, from the mathematics point of view this claim is not a valid one. We accept that Nicodemus talks in terms of whole numbers so that he can say that the next smaller is 1/6. He was based on the fact that the bigger the denominator the smaller the fraction. His next step was to choose the second fraction of the pair having as its denominator the number 10. He justified his choice by saying that it is easy to double this denominator as many times as he wants checking at the same time the equilibrium of the bar on the top of the screen. Thus he started with 1/10, and then he used 1/20 and then 1/40 receiving each time feedback from the program concerning the equilibrium of the upper bar. The first two showed that he had to continue increasing the denominator to obtain equilibrium. However, the final one showed that he outweighed the correct total. Consequently it was time to make a correction by choosing a fraction between 1/20 and 1/40. The 1/30 gave the solution (Figure 2).

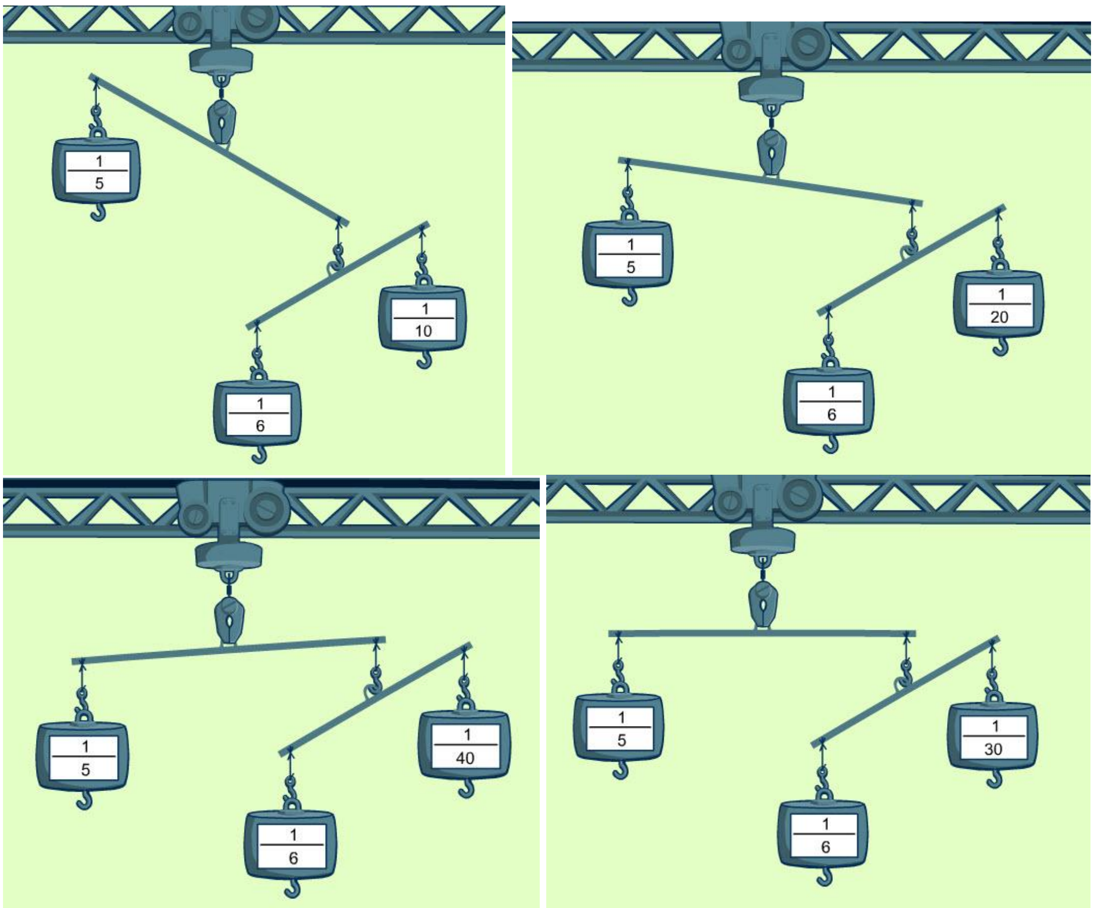

Figure 2. Systematic experimentation

He wrote in his notebook the equation $\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$.

Following the same way of experimenting (i.e., finding the first smaller fraction and then keeping it constant and changing gradually the second one until obtaining equilibrium) he turned to the two remaining unit fractions that were used for the first algorithm. For ½ was more easy to find that $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$.

For 1/9 he repeated the same pattern as in 1/5. He found the first smaller (i.e., 1/10) and then started to change the denominator of the second fractions following the sequence 20, 40, 60, 80, 100. It was between 80 and 100 when the *Balance* showed that he overweighed the total. He corrected by choosing a unit fraction in the middle between 1/80 and 1/100 (i.e., 1/90) which gave the correct answer: $\frac{1}{9} = \frac{1}{10} + \frac{1}{90}$.

Examining the three examples it was easy for him to find that there was a pattern. This helped him to generalize and state his conjecture:

N8: There is a second way to write a unit fraction as sum of two other unit fractions. You can take the next smaller fraction (increasing its denominator by one) and the second fraction will take as denominator the product of the two previous denominators.

Nicodemus verified his algorithm by two additional examples. For each example, he firstly predicted the pair of the unit fractions and then he proceeded to the *Balance* environment to verify his predictions.

Episode Three: Expanding the second algorithm.

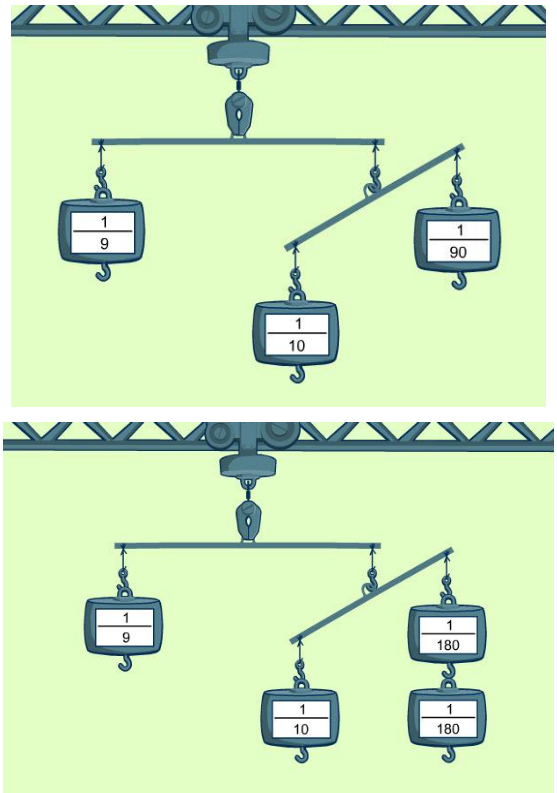

Figure 3. Expanding the second algorithm

After being convinced about his algorithms Nicodemus was asked whether his findings could be used so as to write the fraction 1/9 in his final example as sum of three unit fractions instead of two. His first reaction was to substitute 1/90 with its two halves influenced by his first algorithm. So, his equation became (Figure 3):

$$\frac{1}{9} = \frac{1}{10} + \left(\frac{1}{180} + \frac{1}{180}\right).$$

Immediately, he asked the permission to apply the same algorithm for substituting the 1/10 by its two halves (Figure 4, top). He claimed that this does not influence the equilibrium since according to the first algorithm it is the same to consider 1/10 as 1/20+1/20. He found exciting the idea that he could repeat the algorithm again and he substituted 1/180 by its two halves (Figure 4, bottom):

$$\frac{1}{180} = \frac{1}{360} + \frac{1}{360}.$$

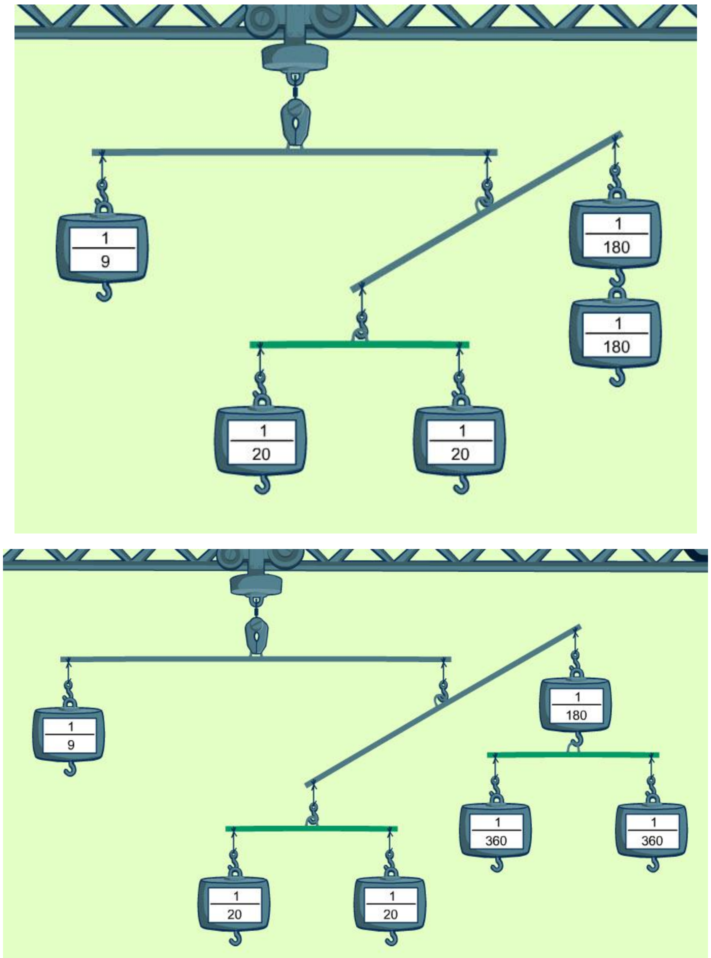

Figure 4. Deeper understanding of the algorithms

It is interesting to point out that Nicodemus was not limited to apply just once more time the first algorithm. He started gradually to suspect that there was something more than the two algorithms.

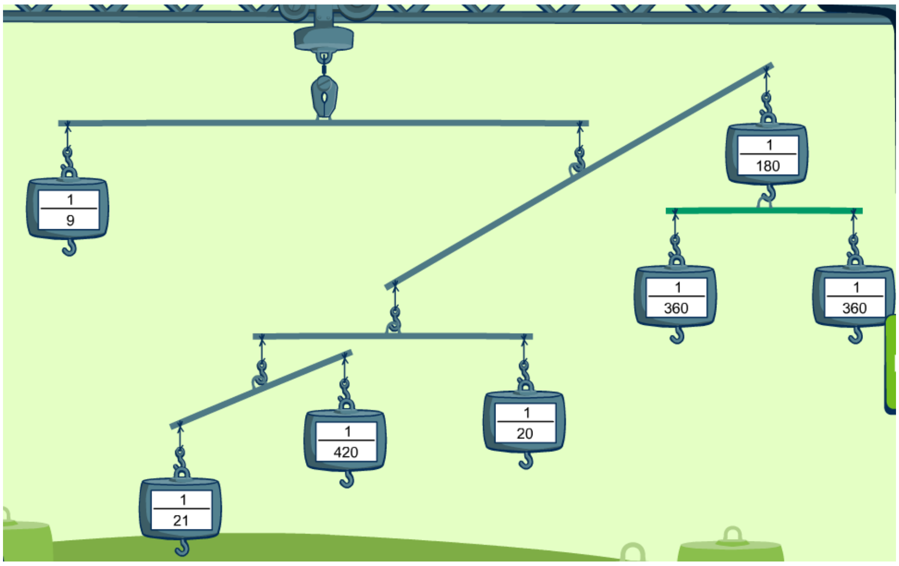

Figure 5. Reaching generalization

He claimed that, since each unit fraction can be written as sum of two other unit fractions this means that it can be applied as many times as he wanted, no matter the side (left or right) of the balance. To show that this is valid for both algorithms he decided to expand his last example by substituting 1/20 with two unit fractions, but now according to the second algorithm (Figure 5):

$$\frac{1}{20} = \frac{1}{21} + \frac{1}{420}$$

The fact is that starting to substitute continually unit fractions with two other unit fractions the denominators gradually were increased and consequently the values of the fractions were decreased. This meant that the change in the slope of the bar on the top of the *Balance* was becoming almost impossible to be noticed. It would be expected that this could cause confusion to the student. However, it is worthy to mention here that during this phase Nicodemus started to ignore the horizontal bar of the balance as a reference to whether the equilibrium had been achieved. He was convinced about the validity of his algorithms and the visual impression was used just to verify his final result rather than the intermediate ones.

At this point he was ready to broaden the initial statement that said that it is possible to find two algorithms that allow each unit fraction to be written as sum of two other unit fractions. After the experimentation that preceded he was able to make a further generalization. The initial statement was correct but could be broaden to become more complete:

N9: Actually, this process can be applied continuously. And any unit fraction can be expanded in a sum of as many unit fractions as we want. The only thing you have to do is to apply one of the two algorithms in order to expand a unit fraction on the right side.

Conclusions

Constructionism as a theory of learning is based on two different notions of construction of knowledge. On the one hand, there is the idea that students learn when they are actively constructing new knowledge rather than waiting for knowledge to be delivered to them. On the other hand, constructionism claims that effective learning takes place when the students are engaged in constructing personally meaningful artefacts which represent their own learning (Beisser, 2006) or when they tinker with an object or entity (Alimissis & Kynigos, 2009). Broadening this perspective the final product of the Nicodemus' reaction with *The Balance* could be considered an instance of an artefact which indeed mirrors his own knowledge as it emerged through his personal engagement in constructing the asked algorithms.

The way such algorithms are usually taught (at least at the level of primary education) is to present the algorithm accompanied by examples and exercises. This leaves students by the impression that an extraordinary mind some time instantly invented the algorithm. This is why it was our decision to give the targeted algorithms as the task that had to be solved. This demands a systematic experimentation and consequently an environment that would allow the student to experiment is necessary. The realistic behaviour and reaction of the *Balance* software contributed to the construction of the second algorithm. The student's engagement in this interplay with the software made him able to improve his attempts, to formulate and check conjectures reaching thus the construction of the algorithm.

Obviously, for a mathematician, being convinced is not enough to accept the validity of this conjecture. It must be followed by an answer to the question why this is true (i.e., to prove it). However, this is not something that is expected from such young students. What Nicodemus found is important in itself. He modified the initial interface of the *Balance* creating a more complex one that constituted the visual description of the algorithm. This was done by adding new components and/or substituting a component by its equivalent pair of fractions.

Obviously, we cannot make generalizations since only one student was involved and this work is considered a case study. But the findings give support to (a) offer another approach to primary school teachers for letting their students interact with an entity in order to construct their own knowledge, and (b) to set up a future research on the same spirit involving now a sufficient number of participants in order to highlight the potential support of certain computer environments in constructing mathematical knowledge.

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