"Metafora" and the fostering of collaborative mathematical problem solving

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Abstract

The learning of problem-solving strategies and heuristics in mathematics has been recognized as of utmost educational importance. Yet, this learning heavily relies on capitalizing on metacognitive abilities which turn the learning of mathematical heuristics to a challenge that involves fostering metacognitive processes. CSCL researchers have posited that collaborative situations and technology-based environments, which allow construction of artefacts and discussion upon them, may support teachers in facilitating small group work in classes. In this paper, we claim that this general approach can be adopted in the case of mathematical problem solving. We show how several teachers used a new platform – the Metafora[1] environment, and how their experience acquired in a workshop, helped them designing activities for their students. This process is exemplified through the case of one teacher and one mathematical challenge designed to foster central mathematical heuristics in collaborative settings. Besides the potentialities of this approach we list major obstacles in this design research program.

Keywords

Mathematical problem-solving, heuristics and learning strategies, collaborative learning, CSCL environments

Learning to solve problems has been defined as a major educational goal in mathematics education (e.g., Schoenfeld, 1992). However, serious obstacles have been detected over the years concerning the ability of teachers to facilitate the learning of problem-solving strategies and heuristics in classroom. The scaffolding idea, according to which teachers' interventions are tuned and calibrated to students needs, is rather complex when it comes to the learning of meta-cognitive competences. The adoption of a pedagogy based on small group learning turns this complexity to an insurmountable challenge. However, CSCL (Computer-Supported Collaborative Learning) tools are intended to facilitate the activity of agents in collaborative contexts. This article presents a platform, and adequate pedagogy, designed to support collaborative mathematical problem-solving by proposing tools for small groups of students in

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[1] http://www.metafora-project.org/; The Metafora project is co-funded by the European Union under the Information and Communication Technologies (ICT) theme of the 7th Framework Programme for R&D (FP7), Contract No. 257872 "

order to solve mathematical problems, learn about their own use of cognitive constructs and to help teachers facilitate group work.

Metafora is an online CSCL system aimed at enabling groups of 2-5 students, 12- to 16-year-old students to participate in inquiry/problem-based activities in science and mathematics, in collaborative settings. Collaboration is a tool for scientific and mathematical activities. However, it is also a goal as students Learn How to Learn Together (L2L2), with their teacher and the software scaffolding. The aim in Metafora is then twofold as students learn scientific and/or mathematical inquiry strategies or heuristics and at the same time, learn general techniques related to collaboration.

Metafora includes a planning/reflecting tool – a shared space with which groups of students collaboratively, and autonomously, construct plans and reflections upon their work. This is being done by a creation of a set of icons called "Visual Language Cards" – a closed set of graphical ontology for scaffolding the construction of on-going plans and reflection on them. The ontology is based on models of inquiry-based learning (e.g. Tamir, 2006), and of problem-solving (e.g. Polya, 1945). As shown in Figure 1, the ontology organizes the collaborative problem solving: Finding hypotheses, simulation, discussion, etc. Also, it serves to monitor actions in order to carry out plan, and to revise the on-going plan in order to adapt to outcomes obtained so far. The visual language also represents scientific/mathematical moves: understanding the problem, reflect, simulate, etc. Naturally, the visual is understood to help gaining control (monitoring and regulating) over actions. From the beginning of the project, the visual language was envisaged to serve as a reflection tool affording students' conscious on-line and post mortem active construction of models of their collaborative mathematical problem/challenge solving process (Hamilton, Lester, Lesh, & Yoon, 2006). Needless to say, the twofold goals aforementioned cannot be attained in a short term period but in a succession of well-designed activities.

In addition to the planning tool, the Metafora platform includes different tools such as LASAD – a graphical tool for facilitating argumentation by the construction of a discussion-map, and other microworlds tailored for specific activities to simulate scientific or mathematical processes
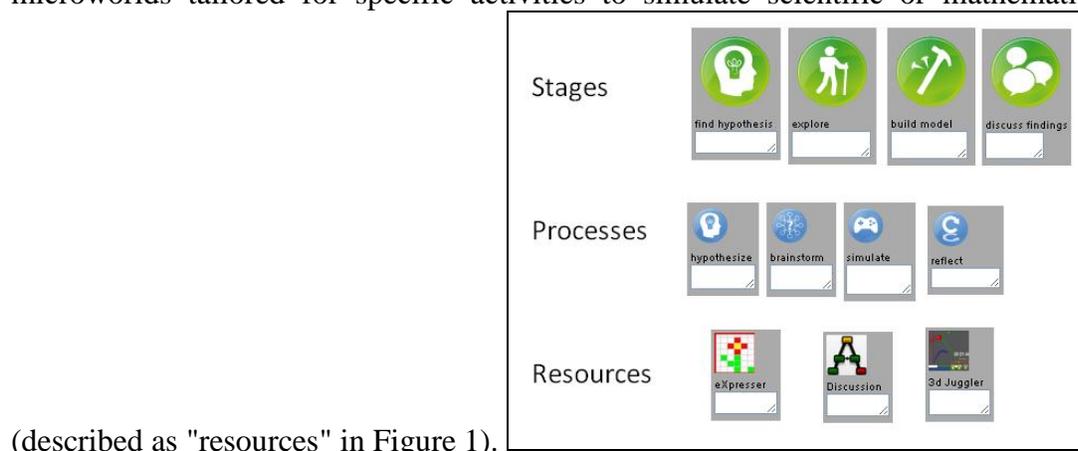

(described as "resources" in Figure 1).

Figure 1. Examples for three of the visual language elements. 1. Stages of the problem solving 2. Processes undertaken during these stages, 3. Resources used to solve the problem
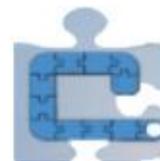
Metafora presents several obstacles to teachers (Abdu, DeGroot & Drachman, 2012). First, its pedagogy is based on challenges – difficult problems with solution processes that is not straightforward, and as such takes more than one lesson to solve. Second, the format of the course – a succession of scientific or mathematical activities, does not lend itself to be easily inserted in existing curricula. Third, teachers are asked to integrate various software when they design the challenges. And last but not the least, students are envisioned to collaborate in small groups, and it is notoriously difficult for teachers to promote and support learning in such settings (Webb, 2009; Schwarz & Asterhan, 2011). In spite of these difficulties, when we advertised our initiative in schools, many teachers showed their interest. Seemingly, a role of mentoring in challenging based activities seemed to some of them more interesting than a role of transmission of normative knowledge.

Methodology

We adopt a *participatory design* methodology in which four science teachers and two mathematics teachers participated in a workshop to prepare themselves to deliver year-long courses in mathematical problem solving and physics. We explained to the teachers the purpose of the design of Metafora and its use. The teachers then solved a challenge in the context of the Metafora environment. Consequently, they adopted a critical approach toward the tool and tried to find ways to improve Metafora for facilitating collaborative problem solving. We videotaped the teachers and transcribed their actions.

In the following sections we present an example for a math challenge, show how one group of teachers solved it in the Metafora environment, and how the experience of one of the mathematics teachers led her to design a problem that is appropriate to the level of her students. More elaborated findings on the development of problem solving heuristics and collaborative heuristics will be reported in further publications.

An example of challenge: The gardener

The Gardener challenge is formulated this way:

> *A gardener wants to create flowerbeds in the form of strips surrounding a central rocky rectangular lot so that (1) the area of the flowerbed is equal to the area of the rocky lot; (2) The central lot and the flowerbed form altogether a rectangular lot whose sides are parallel to those of the rocky lot; (3) the width of the strips is constant and (4) all dimensions (lengths, widths) are integers.*

For junior high-school students as well as for university students, this activity is quite a challenge. The problem is rather open, since while it is relatively easy to find some solutions, it is very difficult to come up with generalized solutions. This challenge provides then many opportunities to implement and/or to learn problem-solving strategies (e.g., Arcavi and Resnick, 2008, Arcavi 1994). For example, in the version that we adopted, no figure is available with the formulation of the challenge, so that sketching a figure becomes a problem solving heuristic (Pólya ,1945) to be learned or to be applied. This absence of figure leads solvers to first understand the problem through by the creation of a sketch of the lot, and to create several examples of flowerbeds. In contrast with school tasks in which variables are already chosen by the designer, representing the challenge algebraically is a heuristic move (Pólya, 1945).

Solving the Gardener in the teacher workshop

In the first step of the study, a teachers' workshop took place in the computer lab. One computer was available for each teacher, but teachers often sat around a common computer. We will now describe the work of a group of teachers, Tsurit, Arnon, and Yael that engaged in the introduction phase and in the Gardener during a session of two hours and half. Their map was constructed progressively while they solved the challenge. Figure 2 displays the outcome that served as a tool for on-going plan and reflection upon the work done.
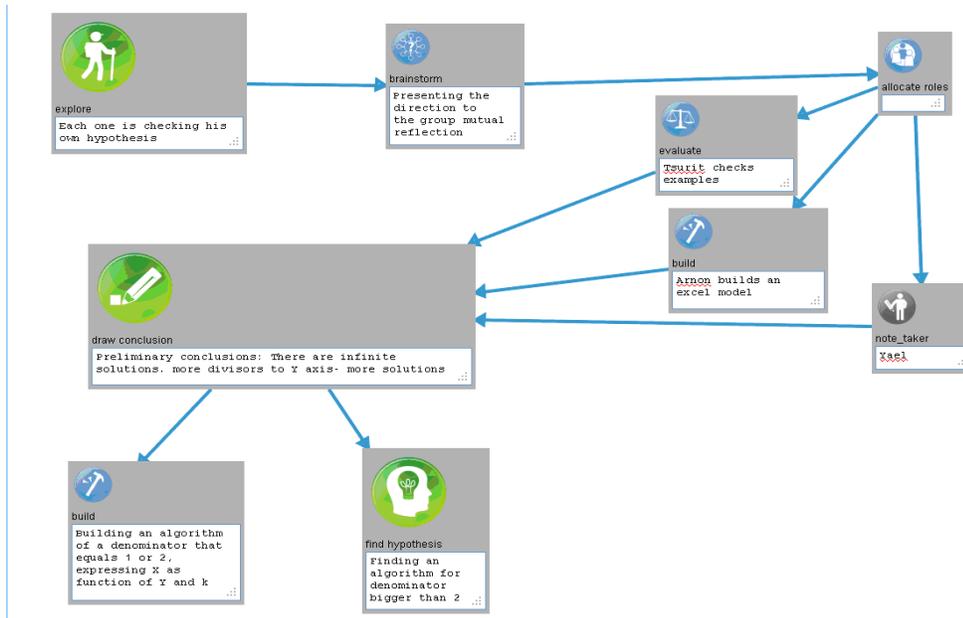

Figure 2 The plan progressively constructed by a group of four teachers

At first, each participant explored the challenge on his own, trying to figure out a possible solution. In order to do so they needed to come to shared understanding. For that matter the teachers co-constructed a mutual figure that serves as a model of a garden (Figure 3). In addition they came to mutual understanding about a formula that represents the relationships between three dimensions of the problem: X, Y and n (Figure 3).

$$X = 2n + \frac{8n^2}{Y-2n}$$

Figure 3 A model of the problem that was created by the solvers, a proper notation and an equation that connects between the three variables.

Then, they gathered around one of the computers for brainstorming. They used the planning tool

to describe what they did so far, so that the planning tool served at this point as a reflecting tool to monitor own actions rather than as a planning tool. Still, the teachers needed to agree on the exact task that they carried together. While doing so we observed two different uses of the cards in the planning-reflecting tool. Yael conveyed a bottom-up approach, as she browsed through the visual symbols list in order to find representations of what she and her colleagues did. For her, the cards served as elicitors for describing the work. Arnon presented a top down approach. His approach was more reflective and abstract, as he first verbalized what the group did, and then tried to find cards that meet with his understanding of what the team did.

The teachers then allocated roles between themselves, and the group split to three parallel venues, based on the three-variables equation (Figure 3): Arnon went to another computer in order to use Excel for building a model for possible solutions. Tsurit decided to work with paper and pencil to create several examples in order to evaluate types and numbers of solutions. Yael – was given the role of a note taker. From that point onward, the planning tool was used by Yael according to Arnon and Tsurit's work. In return, Arnon and Tsurit consulted the on-going plan to decide on further steps. It appeared, from the video of their work that in this stage Yael had difficulties in building the on-going plan. As a result, she received assistance from the teacher in charge of the workshop, in organizing their plan.

Following this individual work, the three teachers gathered in order to draw conclusions from Arnon's Excel patterns and from Tsurit paper and pencil explorations. Arnon created a set of tables for the three variables equation, on one spreadsheet. From this equation, the three teachers reached the conclusion that the number of solutions is infinite. They then focused on their common planning-reflecting map and planned two directions (See figure 2): 1) Building an algorithm to find the values of X, Y and n when the denominator of the formula Y – 2n equals 1 or 2 (then X, Y and n are integers); 2) Finding a hypothesis about other solutions when the denominator Y-2n is bigger than 2. This was the end of their work, since time was up.

The workshop brought several insights, that could serve us in promoting our future work: (1) Elaborating an on-going plan is a demanding task, since (a) students and teachers are hardly familiar with this practice, (b) it requires making a pauses in problem solving, and (c) it capitalizes on the demanding tasks of self-monitoring and self-regulating processes; (2) The planning tool serves first of all to monitor past actions, and this monitoring helps planning further steps in the on-going plan; (3) In spite of all these shortcomings it seems that the planning tool may support the solution of challenges, and more importantly, the learning of collaboration strategies in solving mathematical problems.

Accordingly, we designed a course to foster collaborative problem solving. Next, we show the principles on which the course was based, and how we prepared the implementation of this particular *gardener* scenario.

The Course for fostering collaborative problem solving in Grade 8 students

One of the teachers of the workshop, Tsurit, an experienced teacher in mathematics, decided to organize a course in mathematical problem solving that reaches its end by the time of the writing

of the current lines. Sixteen excellent 8th Grade students met once a week in a 90 min. long session in the computer lab, for eight months. We designed a series of activities to acculturate students to problem solving in small groups. To foster the acculturation to problem solving and collaboration, we (1) we chose problems that encouraged the elaboration of multiple solutions; (2) Like the gardener challenge, some of the problems were open-end challenges, thus affording the elaboration and the application of strategies to solve the challenge (Wee & Looi, 2009); (3) created collaborative situations to trigger productive interactions among groups of students (Dillenbourg, 1999).

The solutions of the challenges differed in their duration from 45 minutes problem solving to 3 weeks challenge collaborative solving. The scenario of the challenge was quite stable:

1. At the beginning of the lesson the teacher presents a challenge to the students, often in a general undetermined way to lead students to see and circumscribe the problem.

2. Groups of students initiate their autonomous work through preliminary explorations. They sometimes use the planning/reflection tool right away to figure out how they envisage solving the challenge. For very challenging tasks, students adopt an approach similar to that of the teachers: Individual work in order to construct preliminary understanding of the challenge, then join forces and reflect upon what they did and plan further solution process.

3. The students solve the challenge; they often turn back to their plan to reflect upon their solution. The teacher passes between the groups, in the class, and supports the different solutions paths. The Metafora maps of the groups serve the teacher to trace mathematical problem solving moves and to propose help.

4. At the end of the process the group recapitulate the work done and reflect upon it. In most cases, when time allows, several groups present their solution processes and their reflections to their classmates.

Sequence of activities: In the background of the course, a design research approach (Cobb et. al., 2004; Mor, 2011) was adopted, according to which the learning environment was assessed and refined and the design of activities became more precise. The course lasted 8 months. It included three main phases:

1. Warming-up activities to instil norms of collaboration in small groups

2. Enculturation to collaborative problem-solving: learning of specific heuristics and strategies; familiarization with the Planning/reflection tool, Geogebra & micro-worlds.

3. Solving challenges with increasing complexity

The course focuses on the following heuristics and strategies: Planning, Reflecting, "Thinking outside the box", abduction (backward strategies), introducing proper notations, Creating a model, Allocating tasks, Generalizing, Checking a simpler case, Hypothesizing, Checking hypotheses, Trial and error, looking for patterns, etc. In addition to the course, we elaborated a preliminary and a closing phase during which students solve problems individually and in groups, to assess development of heuristics and learning strategies in individuals and groups.

It appears that a crucial step for the success of this course was the involvement of the teacher in the design of the challenges. In the following section we will sketch how Tsurit designed with us "the Gardener" challenge, for this particular class. In particular, we undertook an epistemological analysis – meaning that we envisaged possible (complete, partial or flawed) solution paths to be capitalized on during class work.

The design of the Gardener challenge in the framework of Metafora

The teacher needs to be highly prepared for the support of a reflective, computer based, collaborative problem solution processes in multiple parallel groups. Tsurit's challenges were three, then. First, she needed to be familiar with reasonable solution paths with possible milestones, in order to be able to recognize particular solution process (partly through their Metafora maps). Second she needed to come up with an appropriate support that the team might need, in both group and mathematical levels. Third, she needed to be familiar with possible software and environments that might support the solutions.

We will now illustrate two envisaged solution paths that involve two computerized tools, Excel and Geogebra. We start with Tsurit's envisaged presentation of the challenge. Then we will bring an envisioned beginning of the solution by the students. Then we will describe the two possible paths to the solution, based on the solution of the teachers that is described above and solutions of the members of our team. In addition, we will list support she might need to provide.

Tsurit will start the following collaborative script, after the presentation of the problem:

> You have two weeks from now to work on the challenge I will present to you. Within two weeks, each team will present its solution, and will describe the solution process with the help of the planning-reflecting tool. Each team will have 15 minutes for the presentation. I suggest that at first, each one will sit with himself and think about the challenge. After you will gain some sense of the problem, log in to Metafora and plan how you are going to solve the problem. You may use paper and pencil, or any computer simulation you want that might support your work.

Tsurit will then present the challenge. She will then instigate an activity to *Understand the problem*. She will encourage as many questions as possible questions regarding the boundaries of the challenge. She will ask students to first discuss what the challenge is about: "Is there one solution?", "Should we find all solutions?", "Can the rectangle be a square?" No figure will be provided, and the *Drawing of a sketch* by all groups is likely to provide opportunities to understand the problem. The students are familiar with Geogebra, and may use it, or simply use paper and pencil. We envisage that Tsurit will ask one or two groups of students, after a while, to present the challenge and to explain why this is a challenge. We foresee that the groups she will invite will present a sketch and will report that they found several solutions but that they do not know whether they have them all.After this introductory phase, Tsurit will invite students to solve the challenge. We present here two planning/reflection maps that represent two possible solution paths of two imaginary teams.

The first map (Figure 4) is constructed by two students: Misha and Martin. First they follow Tsurit's preliminary scenario, as both of them (1) engage in understanding the problem, and (2) create the sketch of the problem similar to the sketch in Figure 3. Then, they (3) engage in a discussion in which each of them presents his path for the solution. They (4) come up with agreed notations for the sketch, in order to "speak in the same language" (A sketch in which all

variables are mentioned like in Figure 3). We envision that at this stage, Tsurit will offer them to reflect upon their solution so far, and to create a plan onward of their further common work. The students then come up with the explanation of their solution process (Cards 1 to 4) and then plan ahead (cards 5-7). First they (5) allocate tasks. (6) Misha uses Excel in order to come up with possible solutions; the way to achieve this goal is unknown. (7) Martin plans to build a mathematical equation. They both leave the planning tool to come up with solutions. Martin's attempts are successful as he provides an equation $X = 2n(Y+2n)/(Y-2n)$ that connects between the three variables agreed upon. Martin immediately reports on the equation he found with the planning/reflection tool. Misha's first attempts with Excel yield one isolated solution (e.g., $X = 6$, $Y= 4$, $n = 1$). Based on a prompt by Tsurit, Martin is asked to share his knowledge with Misha.
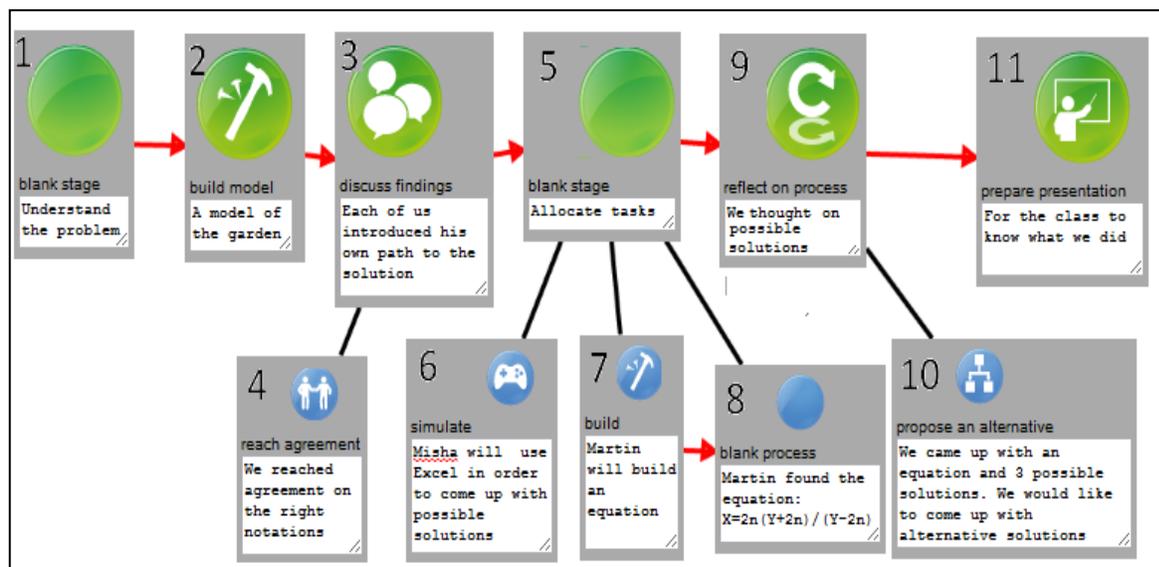

Figure 4: First example of a planning/reflecting map

Martin approaches Misha and shows him the equation and convinces him that Excel will give isolated solutions only. They abandon Excel and (9) try together to find solutions to the equation. At this stage, Tsurit reminds them that the solutions should be natural numbers. The team finds three solutions for the challenge by using trial and error attempts with eth equation, and goes to the planning/reflecting tool to report about it. While doing so, Tsurit asks them if they can generalize their solution. The team then goes back to the map and reports that they would like to (10) come up with alternative solutions. They keep on trying to generalize their ideas, but reach a dead-end. At this point they go back to the map, explain that their next step would be to (11) present their solution to the class.

The second map (Figure 5) is constructed by two other imaginary students, Ofer and Shay. Like Misha and Martin they go through the first two steps (1, 2). They first adopt a trial and error strategy (3). They are then prompted by Tsurit to reflect upon their work and to devise a plan of their future work. They put cards (1-3) on the map. They now decide to adopt more systematic steps in their trial and error attempts. (4) Shay assigns himself to look for software that might help them. After a short discussion they realize that Geogebra might help them. They come up

with an hypothesis (5), according to which if they will find an equation that links between variables, they might be able to display its graph with a grid of natural numbers with Geogebra. Ofer (6) decides to check the simpler case of a square garden. After some on paper computations, he does not manage to find any solution. The team then goes back to the map and report about the unsuccessful trials. They are now stuck. The students (8) reflect upon their work and report that they need to come up with an equation that will link the variables of the model. They (9) create a mathematical model of the problem, based on support that is given to them by Tsurit, similar to the one that was given by the first team: $X = 2n (Y+2n)/(Y-2n)$. When they have the equation, i.e. mathematical model, (11) they ask themselves, if their accomplishment will lead them to a solution. In order to do so they divide again, as (12) Shay start with a simpler case, checking what happens if n=1. For that matter he uses the dynamic geometry software Geogebra. We see in figure 6 an illustration in Geogebra for the function that applies for n=1. Shay finds out that the only integer solution in this graph is a 4X6 rectangle.

Figure 5: Second example of a planning/reflecting map

Ofer (13) goes back to his initial idea in which he verifies what happens when the shape is a square. He uses the mathematical model and places X as equal to Y. His computations lead him
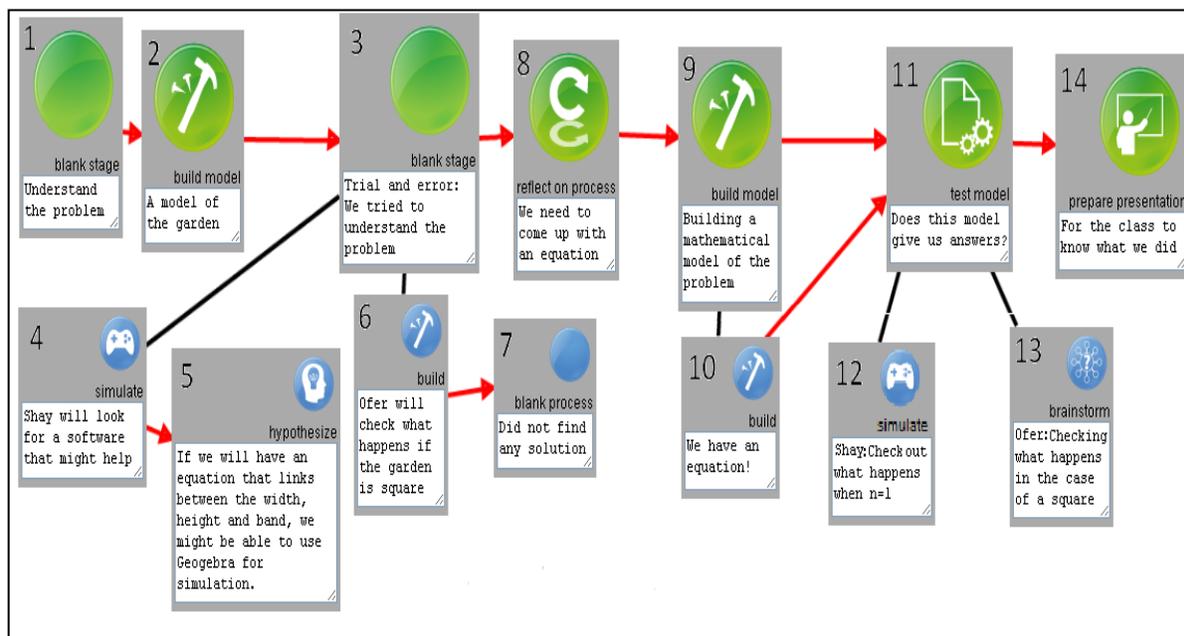
to the following equation: $X = 2n \pm n \cdot \sqrt{8}$. He observes this equation and realizes that if n is an integer, X cannot be an integer, and vica-versa. Last, the team members come up with a way to present their results, and solution process, to the class (14).
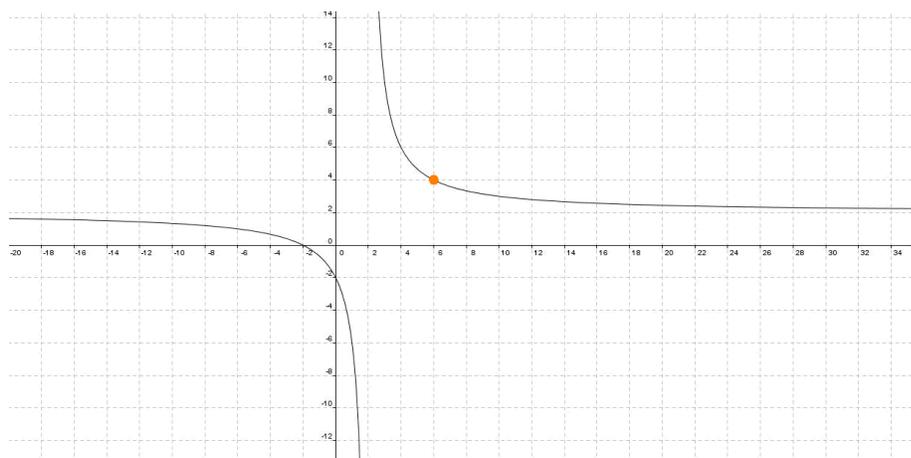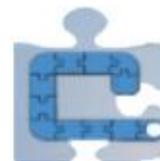
Figure 6: A Geogebra simulation for the function that applies with n=1 which is X=(Y+2)/(Y-2).

The final activity that takes place is the class Reflection in which the students will reflect upon their solution process in front of the class, presenting both their results and their solution process.

Conclusions

The solution paths presented here suggest that students will find some solutions but that they are far from having completed the solution of the challenge (in fact, it can be reduced to a Diophantine equation). As many times during the course, the partial successes of each team in their collaborative work and their difficulties in completing the task will prepare the ground for the teacher's modelling and scaffolding of more sophisticated heuristic moves. In our case, they will be probably mediated through a class reflection upon the various solution paths given by the teams, and the articulation of advanced moves (Checking the equation for X=Y or for n=1, using Geogebra or manipulating the relation $X = 2n * (Y + 2n)/(Y - 2n)$ to obtain $X = 2n + 8n^2/(Y - 2n)$ – leading to the generation of families of solutions by the method of exhaustion). These new moves are learnable because groups of students are now convinced of their necessity. Our paper has shown then that the learning of mathematical heuristics and strategies seems feasible in a CSCL context. We showed that the investment of the teacher in this endeavour is enormous but it seems worthwhile.

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Appendix 1: Possible support that might be given by the teacher

Group learning support:

1. What are you doing?

2. Why are you doing this?

3. How success in this direction could lead you to solve the problem?

4. Are you working according to your plan?

5. Do you want to revise your plan to show what you are doing?

6. Consider comparing your separate work.

7. Have you started working on your activity?

8. Consider asking for help from others.

9. Is this time to revise your plan?

10. Don't forget to reflect on your plan.

11. Does everybody know what he does?

Math challenge support:

1. Does everybody understand the challenge?

2. I suggest that you will draw a sketch of the problem

3. Are you all using the same notation?

4. What is the role of integer numbers in the solution?

5. Do you have a mathematical model?

6. Will modifying the mathematical model help you in this case?

7. You should explore patterns of solutions

8. What other patterns can you find here?

9. You can check the solution for n=1, 2, 3

10. Did you try any computer simulation that might help?

11. What other solutions can you find for this problem?